V

$$AD \cdot DB + \overline{CD}^2 = \overline{CB}^2$$

A E C D B

But $CB = \frac{1}{2} AB = \frac{1}{2}$ of AD DB

Cut off $AE = DB$

Then EC will $= CD$ $\therefore CD = \frac{1}{2} ED = \frac{1}{2}$ AD AE

$= \frac{1}{2}(AD - DB)$

VI

$$AC \cdot CB + \overline{BD}^2 = \overline{DC}^2$$

E A D B C

But $BD = \frac{1}{2} AB = \frac{1}{2}(AC - BC)$

Produce CA to E making $AE = BC$

Then $EA = BC$ & $AD = BD$

$\therefore ED = DC$ $\therefore ED = \frac{1}{2} EC = \frac{1}{2}(AC + EA)$

$= \frac{1}{2}(BC + AC)$

A C B

Let AB be divided into any 2 parts in pt C

Then $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2 + 2 AB \cdot BC$ II 7

or $\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2 AB \cdot BC$ Ax 3

$\therefore (AB - BC)^2 = \overline{AB}^2 + \overline{BC}^2 - 2 AB \cdot BC$

But $(AB + BC)^2 = \overline{AB}^2 + \overline{BC}^2 + 2 AB \cdot BC$ II 4

$\therefore (AB + BC)^2 + (AB - BC)^2 = 2(\overline{AB}^2 + \overline{BC}^2)$ Ax 2

$\therefore$ The sq on the sum of 2 st lines together with the sq on their diff is equal to twice the sum of the sqs on 2 st lines

& this enunciation includes both II 9 & II 10 as may be shown as follows

A C D B

Let st line AB be divided equally in C & unequally in D

Then $\overline{AD}^2 + \overline{BD}^2 = 2(\overline{AC}^2 + \overline{CD}^2)$ II 9

$\therefore (AC + CD)^2 + (BC - CD)^2 = 2(\overline{AC}^2 + \overline{CD}^2)$

$\therefore (AC + CD)^2 + (AC - CD)^2 = 2(\overline{AC}^2 + \overline{CD}^2)$

$\therefore$ The sq on the sum of 2 st lines together with the sq on their diff is equal &c

M.

EL

Of

Pres

Auth

Reco

Reco

Aut

W. J. Gage & Co.'s Mathematical Series.

ELEMENTS OF GEOMETRY.

CONTAINING

BOOKS I. TO III.

WITH

EXERCISES AND NOTES

BY

J. HAMBLIN SMITH, M. A.,

Of Gonville and Caius College, and late Lecturer at St. Peter's College, Cambridge.

Prescribed by the Council of Public Instruction for use in the Schools of Nova Scotia.

Authorized for use in the Schools of Manitoba.

Recommended by the University of Halifax, Nova Scotia.

Recommended by the Council of Public Instruction, Quebec.

Authorized by the Education Department, Ontario.

1886.

W. J. GAGE & COMPANY,

TORONTO.

A C B D

Let st line AB be bisected in pt C & produced to D

then $\overline{AD}^2 + \overline{BD}^2 = 2(\overline{AC}^2 + \overline{CD}^2)$ II 10

$\therefore (AC+CD)^2 + (CD-CB)^2 = 2(\overline{AC}^2 + \overline{CD}^2)$

$\therefore (AD+AC)^2 + (CD-AC)^2 = 2(\overline{CD}^2 + \overline{AC}^2)$

Wherefore the sq on the sum of st lines

Hence II 9 & II 10 might be superseded by II 4, II 7 & Ax 2 & 3.

3500

PREFACE.

To preserve Euclid's order, to supply omissions, to remove defects, to give brief notes of explanation and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the Elements.

The work is based on the Greek text, as it is given in the Editions of August and Peyrard. To the suggestions of the late Professor De Morgan, published in the Companion to the British Almanack for 1849, I have paid constant deference.

A limited use of symbolic representation, wherein the symbols stand for words and not for operations, is generally regarded as desirable, and I have been assured, by the highest authorities on this point, that the symbols employed in this book are admissible in the Examinations at Oxford and Cambridge.[1]

I have generally followed Euclid's method of proof, but not to the exclusion of other methods recom-

[1] I regard this point as completely settled in Cambridge by the following notices prefixed to the papers on Euclid set in the Senate-House Examinations:

I. In the Previous Examination:

In answers to these questions any intelligible symbols and abbreviations may be used.

II. In the Mathematical Tripos:

In answers to the questions on Euclid the symbol — must not be used. The only abbreviation admitted for the square on AB *is* "*sq. on* AB," *and for the rectangle contained by* AB *and* CD, "*rect.* AB, CD."

mended by their simplicity, such as the demonstrations by which I propose to replace (at least for a first reading) the difficult Theorems 5 and 7 in the First Book. I have also attempted to render many of the proofs, as for instance Propositions 2, 13, and 35 in Book I., and Proposition 13 in Book II., less confusing to the learner.

In Propositions 4, 5, 6, 7, and 8 of the Second Book I have ventured to make an important change in Euclid's mode of exposition, by omitting the diagonals from the diagrams and the gnomons from the text.

In the Third Book I have deviated with even greater boldness from the precise line of Euclid's method. For it is in treating of the properties of the circle that the importance of certain matters, to which reference is made in the Notes of the present volume, is fully brought out. I allude especially to the application of Superposition as a test of equality, to the conception of an Angle as a magnitude capable of unlimited increase, and to the development of the methods connected with Loci and Symmetry.

The Exercises have been selected with considerable care, chiefly from the Senate House Examination Papers. They are intended to be progressive and easy, so that a learner may from the first be induced to work out something for himself.

I desire to express my thanks to the friends who have improved this work by their suggestions, and to beg for further help of the same kind.

J. HAMBLIN SMITH

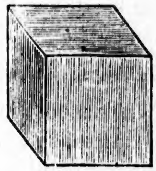
CAMBRIDGE, 1873.

ELEMENTS OF GEOMETRY.

INTRODUCTORY REMARKS.

WHEN a block of stone is hewn from the rock, we call it a Solid *Body.* The stone-cutter shapes it, and brings it into that which we call *regularity of form;* and then it becomes a Solid *Figure.*

Now suppose the figure to be such that the block has six flat sides, each the exact counterpart of the others ; so that, to one who stands facing a corner of the block, the three sides which are visible present the appearance represented in this diagram.



Each side of the figure is called a *Surface* ; and when smoothed and polished, it is called a *Plane* Surface.

The sharp and well-defined edges, in which each pair of sides meets, are called *Lines.*

The place, at which any three of the edges meet, is called a *Point.*

A *Magnitude* is anything which is made up of parts in any way like itself. Thus, a line is a magnitude ; because we may regard it as made up of parts which are themselves lines.

The properties Length, Breadth (or Width), and Thickness (or Depth or Height) of a body are called its *Dimensions.*

We make the following distinction between Solids, Surfaces Lines, and Points :

A Solid has three dimensions, Length, Breadth, Thickness.

A Surface has two dimensions, Length, Breadth.

A Line has one dimension, Length.

A point has no dimensions.

BOOK I.

DEFINITIONS.

I. A **Point** is that which has no parts.

This is equivalent to saying that a Point has no magnitude, since we define it as that which cannot be divided into smaller parts.

II. A **Line** is length without breadth.

We cannot conceive a visible line without breadth; but we can reason about lines as if they had no breadth, and this is what Euclid requires us to do.

III. The **Extremities** of finite **Lines** are points.

A point marks *position*, as for instance, the place where a line begins or ends, or meets or crosses another line.

IV. A **Straight Line** is one which lies in the same direction from point to point throughout its length.

V. A **Surface** is that which has length and breadth only.

VI. The **Extremities** of a **Surface** are lines.

VII. A **Plane Surface** is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

Thus the ends of an uncut cedar-pencil are plane surfaces; but the rest of the surface of the pencil is not a plane surface, since two points may be taken in it such that the *straight* line joining them will not lie on the surface of the pencil.

In our introductory remarks we gave examples of a Surface, a Line, and a Point, as we know them through the evidence of the senses.

2

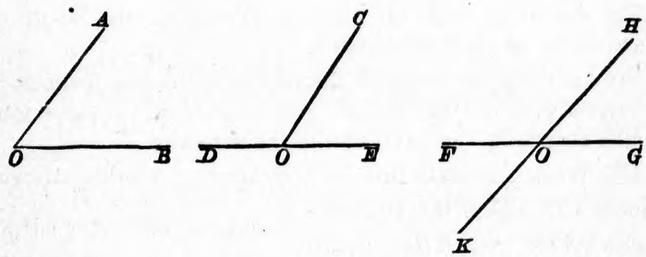
The Surfaces, Lines, and Points of Geometry may be regarded as mental pictures of the surfaces, lines, and points which we know from experience.

It is, however, to be observed that Geometry requires us to conceive the possibility of the existence

of a Surface apart from a Solid body,
of a Line apart from a Surface.
of a Point apart from a Line.

VIII. When two straight lines meet one another, the inclination of the lines to one another is called an ANGLE.

When *two* straight lines have one point common to both, they are said to *form* an angle (or angles) at that point. The point is called the *vertex* of the angle (or angles), and the lines are called the *arms* of the angle (or angles).

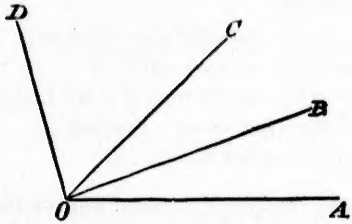


Thus, if the lines *OA*, *OB* are terminated at the same point *O*, they form an angle, which is called *the angle at O*, or *the angle AOB*, or *the angle BOA*,—the letter which marks the vertex being put between those that mark the arms.

Again, if the line *CO* meets the line *DE* at a point in the line *DE*, so that *O* is a point common to both lines, *CO* is said to make with *DE* the angles *COD*, *COE*; and these (as having one arm, *CO*, common to both) are called *adjacent* angles.

Lastly, if the lines *FG*, *HK* cut each other in the point *O*, the lines make with each other four angles *FOH*, *HOG*, *GOK*, *KOF*; and of these *GOH*, *FOK* are called *vertically opposite* angles, as also are *FOH* and *GOK*.

When *three or more* straight lines as OA, OB, OC, OD have a point O common to all, the angle formed by one of them, OD,



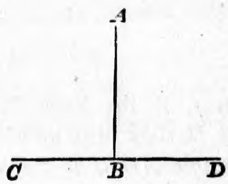
with OA may be regarded as being made up of the angles AOB, BOC, COD; that is, we may speak of the angle AOD as a whole, of which the parts are the angles AOB, BOC, and COD.

Hence we may regard an angle as a *Magnitude*, inasmuch as any angle may be regarded as being made up of parts which are themselves angles.

The size of an angle depends in no way on the length of the arms by which it is bounded.

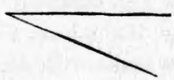
We shall explain hereafter the restriction on the magnitude of angles enforced by Euclid's definition, and the important results that follow an extension of the definition.

IX. When a straight line (as AB) meeting another straight line (as CD) makes the adjacent angles (ABC and ABD) equal to one another, each of the angles is called a RIGHT ANGLE; and each line is said to be a PERPENDICULAR to the other.



X. An OBTUSE ANGLE is one which is greater than a right angle.

XI. An ACUTE ANGLE is one which is less than a right angle.

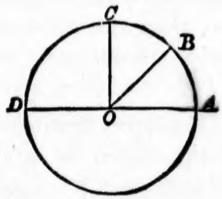


XII. A FIGURE is that which is enclosed by one or more boundaries.

XIII. A Circle is a plane figure contained by one line, which is called the Circumference, and is such, that all straight lines drawn to the circumference from a certain point (called the Centre) within the figure are equal to one another.

XIV. Any straight line drawn from the centre of a circle to the circumference is called a Radius.

XV. A Diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference.



Thus, in the diagram, O is the centre of the circle $ABCD$, OA, OB, OC, OD are Radii of the circle, and the straight line AOD is a Diameter. Hence the radius of a circle is half the diameter.

XVI. A Semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XVII. Rectilinear figures are those which are contained by straight lines.

The Perimeter (or Periphery) of a rectilinear figure is the sum of its sides.

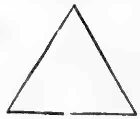
XVIII. A Triangle is a plane figure contained by three straight lines.

XIX. A Quadrilateral is a plane figure contained by four straight lines.

XX. A Polygon is a plane figure contained by more than four straight lines.

When a polygon has all its sides equal and all its angles equal it is called a *regular* polygon.

XXI. An Equilateral Triangle is one which has all its sides equal.



XXII. An Isosceles Triangle is one which has two sides equal.

The third side is often called the *base* of the triangle.

The term *base* is applied to any one of the sides of a triangle to distinguish it from the other two, especially when they have been previously mentioned.

XXIII. A Right-angled Triangle is one in which one of the angles is a right angle.

The side *subtending*, that is, *which is opposite* the right angle, is called the *Hypotenuse*.

XXIV. An Obtuse-angled Triangle is one in which one of the angles is obtuse.

It will be shewn hereafter that a triangle can have only one of its angles either equal to, or greater than, a right angle.

XXV. An Acute-angled Triangle is one in which all the angles are acute.

XXVI. Parallel Straight Lines are such as, being in the same plane, never meet when continually produced in both directions.

Euclid proceeds to put forward Six Postulates, or Requests, that he may be allowed to make certain assumptions on the construction of figures and the properties of geometrical magnitudes.

Postulates

Let it be granted—

I. That a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. That a circle may be described from any centre at any distance from that centre.

IV. That all right angles are equal to one another.

V. That two straight lines cannot enclose a space.

VI. That if a straight line meet two other straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced shall at length meet upon that side, on which are the angles, which are together less than two right angles.

The word rendered "Postulates" is in the original *αἰτήματα*, "requests."

In the first three Postulates Euclid states the use, under certain restrictions, which he desires to make of certain instruments for the construction of lines and circles.

In Post. I. and II. he asks for the use of the straight ruler, wherewith to draw straight lines. The restriction is, that the ruler is not supposed to be marked with divisions so as to measure lines.

In Post. III. he asks for the use of a pair of compasses, wherewith to describe a circle, whose centre is at one extremity of a given line, and whose circumference passes through the other extremity of that line. The restriction is, that the compasses are not supposed to be capable of conveying distances.

Post. IV. and V. refer to simple geometrical facts, which Euclid desires to take for granted.

Post. VI. may, as we shall shew hereafter, be deduced from a more simple Postulate. The student must defer the consideration of this Postulate, till he has reached the 17th Proposition of Book I.

Euclid next enumerates, as statements of fact, nine Axioms

or, as he calls them, Common Notions, applicable (with the exception of the eighth) to all kinds of magnitudes, and not necessarily restricted, as are the Postulates, to *geometrical* magnitudes.

Axioms.

I. Things which are equal to the same thing are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals and unequals be added together, the wholes are unequal.

V. If equals be taken from unequals, or unequals from equals, the remainders are unequal.

VI. Things which are double of the same thing, or of equal things, are equal to one another.

VII. Things which are halves of the same thing, or of equal things, are equal to one another.

VIII. Magnitudes which coincide with one another are equal to one another.

IX. The whole is greater than its part.

With his Common Notions Euclid takes the ground of authority, saying in effect, "To my Postulates I request, to my Common Notions I claim, your assent."

Euclid develops the science of Geometry in a series of Propositions, some of which are called Theorems and the rest Problems, though Euclid himself makes no such distinction.

By the name *Theorem* we understand a truth, capable of demonstration or proof by deduction from truths previously admitted or proved.

By the name *Problem* we understand a construction, capable of being effected by the employment of principles of construction previously admitted or proved.

A *Corollary* is a Theorem or Problem easily deduced from, or effected by means of, a Proposition to which it is attached.

We shall divide the First Book of the Elements into three sections. The reason for this division will appear in the course of the work.

SYMBOLS AND ABBREVIATIONS USED IN BOOK I.

$\because$ *for*	because	$\odot$ *for*	circle
$\therefore$	therefore	$\bigcirc$ce............	circumference
$=$	is (or are) equal to	$\parallel$	parallel
$\angle$	angle	▱	parallelogram
$\triangle$	triangle	$\perp$	perpendicular

equilat.	equilateral	reqd.	required
extr..........	exterior	rt...............	right
intr..........	interior	sq.	square
pt	point	sqq.............	squares
rectil.	rectilinear	st...............	straight

It is well known that one of the chief difficulties with learners of Euclid is to distinguish between what is assumed, or given, and what has to be proved in some of the Propositions. To make the distinction clearer we shall put in italics the statements of what has to be done in a Problem, and what has to be proved in a Theorem. The last line in the proof of every Proposition states, that what had to be done or proved has been done or proved.

The letters Q. E. F. at the end of a Problem stand for *Quod erat faciendum*.

The letters Q. E. D. at the end of a Theorem stand for *Quod erat demonstrandum*.

In the marginal references :

Post. stands for	Postulate.
Def.	Definition.
Ax.	Axiom.
I. 1.	Book I. Proposition 1.

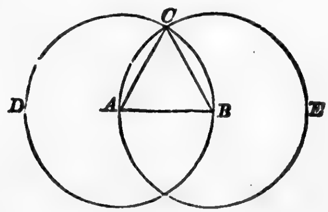
Hyp. stands for Hypothesis, *supposition*, and refers to something granted, or assumed to be true.

SECTION I.

On the Properties of Triangles.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle on a given straight line.



Let AB be the given st. line.

It is required to describe an equilat. △ *on* AB

With centre A and distance AB describe ⊙ BCD. Post. 3.
With centre B and distance BA describe ⊙ ACE. Post. 3.

From the pt. C, in which the ⊙s cut one another, draw the st. lines CA, CB. Post. 1.

Then will ABC be an equilat. △.

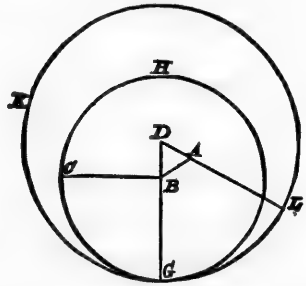
For ∵ A is the centre of ⊙ BCD,
∴ $AC=AB$. Def. 13
And ∵ B is the centre of ⊙ ACE,
∴ $BC=AB$. Def. 13.
Now ∵ AC, BC are each $=AB$,
∴ $AC=BC$. Ax. 1.

Thus AC, AB, BC are all equal, and an equilat. △ ABC has been described on AB.

Q. E. F.

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.



Let A be the given pt., and BC the given st. line.

It is required to draw from A a st. line equal to BC.

From A to B draw the st. line AB. Post. 1.

On AB describe the equilat. $\triangle$ ABD. I. 1.

With centre B and distance BC describe $\odot$ CGH. Post. 3.

Produce DB to meet the $\bigcirc$ce CGH in G.

With centre D and distance DG describe $\odot$ GKL. Post. 3.

Produce DA to meet the $\bigcirc$ce GKL in L.

Then will $AL=BC$.

For $\because$ B is the centre of $\odot$ CGH,

$\therefore$ $BC=BG$. Def. 13.

And $\because$ D is the centre of $\odot$ GKL,

$\therefore$ $DL=DG$. Def. 13.

And parts of these, DA and DB, are equal. Def. 21

$\therefore$ remainder AL=remainder BG. Ax. 3.

But $BC=BG$;

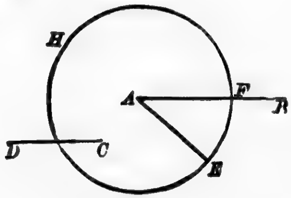
$\therefore$ $AL=BC$. Ax. 1.

Thus from pt. A a st. line AL has been drawn $=BC$.

Q. E. F.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB be the greater of the two given st. lines AB, CD.

It is required to cut off from AB a part$=CD$.

From A draw the st. line $AE=CD$. I. 2.

With centre A and distance AE describe ⊙ EFH, cutting AB in F.

Then will $AF=CD$.

For $\because$ A is the centre of ⊙ EFH,

$\therefore$ $AF=AE$.

But $AE=CD$;

$\therefore$ $AF=CD$. Ax. 1.

Thus from AB a part AF has been cut off$=CD$.

Q. E. F.

EXERCISES.

1. Shew that if straight lines be drawn from A and B in the diagram of Prop. I. to the other point in which the circles intersect, another equilateral triangle will be described on AB.

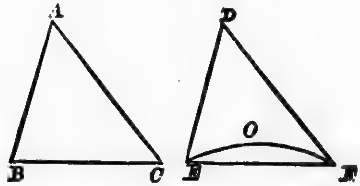
2. By a construction similar to that in Prop. III. produce the less of two given straight lines that it may be equal to the greater.

3. Draw a figure for the case in Prop. II., in which the given point coincides with B.

4. By a similar construction to that in Prop. I. describe on a given straight line an isosceles triangle, whose equal sides shall be each equal to another given straight line.

Proposition IV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they must have their third sides equal; and the two triangles must be equal, and the other angles must be equal, each to each, viz. those to which the equal sides are opposite.



In the $\triangle$s ABC, DEF,
let $AB=DE$, and $AC=DF$, and $\angle\, BAC=\angle\, EDF$.

Then must $BC=EF$ and $\triangle\, ABC=\triangle\, DEF$, and the other $\angle$s, to which the equal sides are opposite, must be equal, that is, $\angle\, ABC=\angle\, DEF$ and $\angle\, ACB=\angle\, DFE$.

For, if $\triangle\, ABC$ be applied to $\triangle\, DEF$,
so that A coincides with D, and AB falls on DE,
then $\because AB=DE$, $\therefore B$ will coincide with E.
And $\because AB$ coincides with DE, and $\angle\, BAC=\angle\, EDF$, Hyp.
$\therefore AC$ will fall on DF.
Then $\because AC=DF$, $\therefore C$ will coincide with F.
And $\because B$ will coincide with E, and C with F,
$\therefore BC$ will coincide with EF;
for if not, let it fall otherwise as EOF: then the two st. lines BC, EF will enclose a space, which is impossible. Post. 5.
$\therefore BC$ will coincide with and $\therefore$ is equal to EF, Ax. 8.
and $\triangle\, ABC$.. $\triangle\, DEF$,
and $\angle\, ABC$.. $\angle\, DEF$,
and $\angle\, ACB$.. $\angle\, DFE$.

Q. E. D.

NOTE 1. *On the Method of Superposition.*

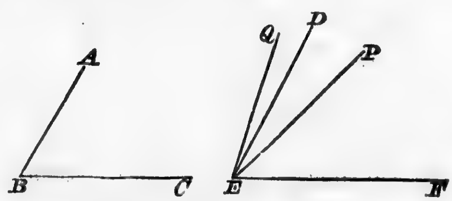
Two geometrical magnitudes are said, in accordance with Ax. VIII. to be *equal*, when they can be so placed that the boundaries of the one coincide with the boundaries of the other.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide: and two angles are equal, if they can be so placed that their vertices coincide in position and their arms in direction: and two triangles are equal, if they can be so placed that their sides coincide in direction and magnitude.

In the application of the test of equality by this *Method of Superposition*, we assume that an angle or a triangle may be moved from one place, turned over, and put down in another place, without altering the relative positions of its boundaries.

We also assume that if one part of a straight line coincide with one part of another straight line, the other parts of the lines also coincide in direction; or, that straight lines, which coincide in two points, coincide when produced.

The method of Superposition enables us also to compare magnitudes of the same kind that are unequal. For example, suppose ABC and DEF to be two given angles.



Suppose the arm BC to be placed on the arm EF, and the vertex B on the vertex E.

Then, if the arm BA coincide in direction with the arm ED, the angle ABC is equal to DEF.

If BA fall between ED and EF in the direction EP, ABC is less than DEF.

If BA fall in the direction EQ so that ED is between EQ and EF, ABC is greater than DEF.

NOTE 2. *On the Conditions of Equality of two Triangles.*

A Triangle is composed of six parts, three sides and three angles.

When the six parts of one triangle are equal to the six parts of another triangle, each to each, the Triangles are said to be equal in all respects.

There are four cases in which Euclid proves that two triangles are equal in all respects; viz., when the following parts are equal in the two triangles.

1. Two sides and the angle between them. I. 4.
2. Two angles and the side between them. I. 26.
3. The three sides of each. I. 8.
4. Two angles and the side opposite one of them. I. 26.

The Propositions, in which these cases are proved, are the most important in our First Section.

The first case we have proved in Prop. IV.

Availing ourselves of the method of superposition, we can prove Cases 2 and 3 by a process more simple than that employed by Euclid, and with the further advantage of bringing them into closer connexion with Case 1. We shall therefore give three Propositions, which we designate A, B, and C, in the Place of Euclid's Props. V. VI. VII. VIII.

The displaced Propositions will be found on pp. 108-112.

Proposition A corresponds with Euclid I. 5.
.... B .... I. 26, first part.
.... C .... I. 8.

Proposition A. Theorem.

If two sides of a triangle be equal, the angles opposite those sides must also be equal.

Fig. 1.

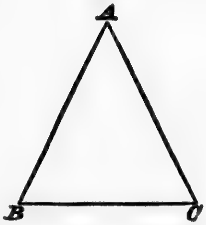


Fig. 2.

In the isosceles triangle ABC, let $AC=AB$. (Fig. 1.)

Then must $\angle ABC = \angle ACB$.

Imagine the $\triangle ABC$ to be taken up, turned round, and set down again in a reversed position as in Fig. 2, and designate the angular points A', B', C'.

Then in $\triangle$s ABC, $A'C'B'$,

$\because AB=A'C'$, and $AC=A'B'$, and $\angle BAC = \angle C'A'B'$,

$\therefore \angle ABC = \angle A'C'B'$. I. 4.

But $\angle A'C'B' = \angle ACB$;

$\therefore \angle ABC = \angle ACB$. Ax. 1.

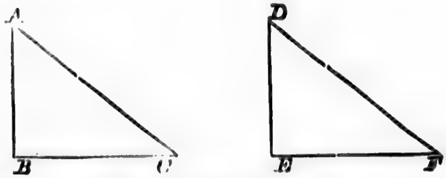
Q.E.D.

Cor. Hence every equilateral triangle is also equiangular.

Note. When one side of a triangle is distinguished from the other sides by being called the *Base*, the angular point opposite to that side is called the *Vertex* of the triangle.

Proposition B. Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $BC = EF$.

Then must $AB = DE$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

For if $\triangle DEF$ be applied to $\triangle ABC$, so that E coincides with B, and EF falls on BC;

then $\because EF = BC$, $\therefore F$ will coincide with C;

and $\because \angle DEF = \angle ABC$, $\therefore ED$ will fall on BA;

$\therefore D$ will fall on BA or BA produced.

Again, $\because \angle DFE = \angle ACB$, $\therefore FD$ will fall on CA;

$\therefore D$ will fall on CA or CA produced.

$\therefore D$ must coincide with A, the only pt. common to BA and CA.

$\therefore DE$ will coincide with and $\therefore$ is equal to AB,

and DF .... AC,

and $\angle EDF$ .... $\angle BAC$,

and $\triangle DEF$ .... $\triangle ABC$;

and $\therefore$ the triangles are equal in all respects.

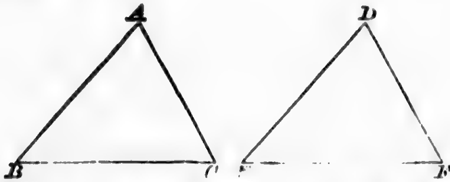
Q. E. D.

Cor. Hence, by a process like that in Prop. A, we can prove the following theorem:

If two angles of a triangle be equal, the sides which subtend them are also equal. (Eucl. I. 6.)

PROPOSITION C. THEOREM.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles must be equal in all respects.

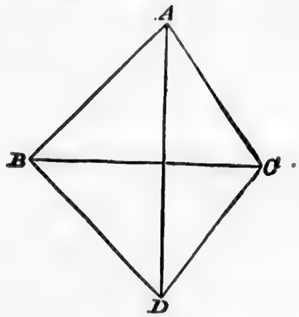


Let the three sides of the $\triangle$s ABC, DEF be equal, each to each, that is, $AB=DE$, $AC=DF$, and $BC=EF$.

Then must the triangles be equal in all respects.

Imagine the $\triangle DEF$ to be turned over and applied to the $\triangle ABC$, in such a way that EF coincides with BC, and the vertex D falls on the side of BC opposite to the side on which A falls; and join AD.

CASE I. When AD passes through BC.



Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$, I. A.

And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD = \angle CDA$, I. A.

$\therefore$ sum of $\angle$s BAD, CAD = sum of $\angle$s BDA, CDA, Ax. 2.

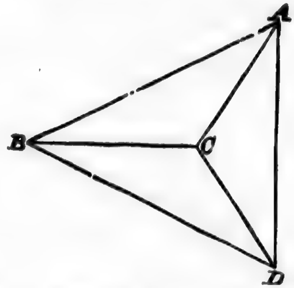
that is, $\angle BAC = \angle BDC$.

Hence we see, referring to the original triangles, that

$\angle BAC = \angle EDF$.

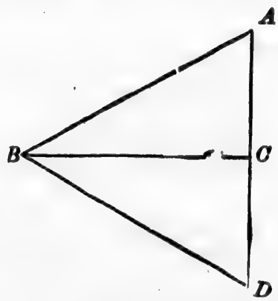
$\therefore$, by Prop. 4, the triangles are equal in all respects.

CASE II. When the line joining the vertices does not pass through BC.



Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$, I. A.
And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD = \angle CDA$, I. A.
Hence since the whole angles BAD, BDA are equal.
and parts of these CAD, CDA are equal.
$\therefore$ the remainders BAC, BDC are equal. Ax. 3.
Then, as in Case I., the equality of the original triangles may be proved.

CASE III. When AC and CD are in the same straight line.



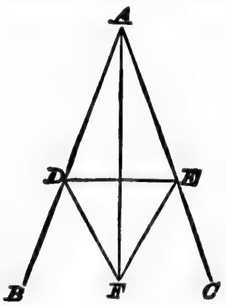
Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$, I. A.
that is, $\angle BAC = \angle BDC$.

Then, as in Case I., the equality of the original triangles may be proved.

Q. E. D.

Proposition IX. Problem.

To bisect a given angle.



Let BAC be the given angle.

It is required to bisect $\angle BAC$.

In AB take any pt. D.

In AC make $AE=AD$, and join DE.

On DE, on the side remote from A, describe an equilat. $\triangle DFE$. I. 1.

Join AF. Then AF will bisect $\angle BAC$.

For in $\triangle$s AFD, AFE,

$\because AD=AE$, and AF is common, and $FD=FE$,

$\therefore \angle DAF=\angle EAF$, I. c.

that is, $\angle BAC$ is bisected by AF.

Q. E. F.

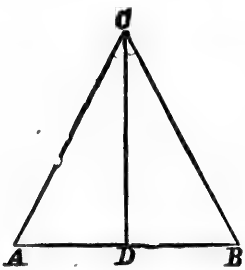
Ex. 1. Shew that we can prove this Proposition by means of Prop. IV. and Prop. A., without applying Prop. C.

Ex. 2. If the equilateral triangle, employed in the construction, be described with its vertex towards the given angle; shew that there is one case in which the construction will fail, and two in which it will hold good.

Note.—The line dividing an angle into two equal parts is called the Bisector of the angle.

Proposition X. Problem.

To bisect a given finite straight line.



Let AB be the given st. line.

It is required to bisect AB.

On AB describe an equilat. $\triangle\, ACB$. I. 1.

Bisect $\angle\, ACB$ by the st. line CD meeting AB in D; I. 9.

then AB shall be bisected in D.

For in $\triangle$s ACD, BCD,

$\because AC=BC$, and CD is common, and $\angle\, ACD = \angle\, BCD$,

$\therefore AD=BD$; I. 4.

$\therefore AB$ is bisected in D.

Q. E. F.

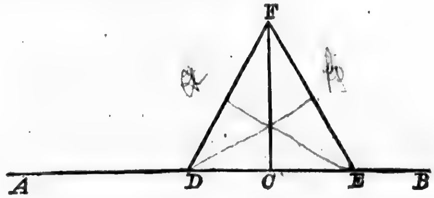
Ex. 1. The straight line, drawn to bisect the vertical angle of an isosceles triangle, also bisects the base.

Ex. 2. The straight line, drawn from the vertex of an isosceles triangle to bisect the base, also bisects the vertical angle.

Ex. 3. Produce a given finite straight line to a point, such that the part produced may be one-third of the line, which is made up of the whole and the part produced.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line from a given point in the same.



Let AB be the given st. line, and C a given pt. in it.

It is required to draw from C a st. line $\perp$ to AB.

Take any pt. D in AC, and in CB make $CE=CD$.

On DE describe an equilat. $\triangle$ DFE. I. 1.

Join FC. FC shall be $\perp$ to AB.

For in $\triangle$s DCF, ECF,

$\because$ $DC=CE$, and CF is common, and $FD=FE$,

$\therefore$ $\angle DCF = \angle ECF$; I. c.

and $\therefore$ FC is $\perp$ to AB. Def. 9.

Q. E. F.

COR. To draw a straight line at right angles to a given straight line AC from one extremity, C, take any point D in AC, produce AC to E, making $CE=CD$, and proceed as in the proposition.

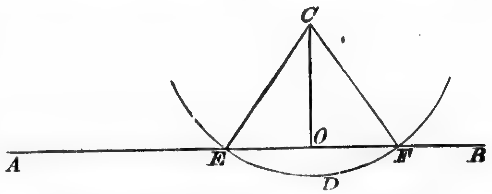
Ex. 1. Shew that in the diagram of Prop. IX. AF and ED intersect each other at right angles, and that ED is bisected by AF.

Ex. 2. If O be the point in which two lines, bisecting AB and AC, two sides of an equilateral triangle, at right angles, meet; shew that OA, OB, OC are all equal.

Ex. 3. Shew that Prop. XI. is a particular case of Prop. IX.

Proposition XII. Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.



Let AB be the given st. line of unlimited length; C the given pt. without it.

It is required to draw from C a st. line $\perp$ to AB.

Take any pt. D on the other side of AB.

With centre C and distance CD describe a $\odot$ cutting AB in E and F.

Bisect EF in O, and join CE, CO, CF. I. 10

Then CO shall be $\perp$ to AB.

For in $\triangle$s COE, COF,

$\because$ $EO=FO$, and CO is common, and $CE=CF$,

$\therefore$ $\angle COE = \angle COF$; I. c.

$\therefore$ CO is $\perp$ to AB. Def. 9.

Q. E. F.

Ex. 1. If the straight line were not of unlimited length, how might the construction fail?

Ex. 2. If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.

Ex. 3. The lines drawn from the angular points of an equilateral triangle to the middle points of the opposite sides are equal.

Miscellaneous Exercises on Props. I. to XII.

1. Draw a figure for Prop. II. for the case when the given point A is

(α) below the line BC and to the right of it.
(β) below the line BC and to the left of it.

2. Divide a given angle into four equal parts.

3. The angles B, C, at the base of an isosceles triangle, are bisected by the straight lines BD, CD, meeting in D; shew that BDC is an isosceles triangle.

4. D, E, F are points taken in the sides BC, CA, AB, of an equilateral triangle, so that $BD=CE=AF$. Shew that the triangle DEF is equilateral.

5. In a given straight line find a point equidistant from two given points; 1st, on the same side of it; 2d, on opposite sides of it.

6. ABC is a triangle having the angle ABC acute. In BA, or BA produced, find a point D such that $BD=CD$.

7. The equal sides AB, AC, of an isosceles triangle ABC are produced to points F and G, so that $AF=AG$. BG and CF are joined, and H is the point of their intersection. Prove that $BH=CH$, and also that the angle at A is bisected by AH.

8. BAC, BDC are isosceles triangles, standing on opposite sides of the same base BC. Prove that the straight line from A to D bisects BC at right angles.

9. In how many directions may the line AE be drawn in Prop. III.?

10. The two sides of a triangle being produced, if the angles on the other side of the base be equal, shew that the triangle is isosceles.

11. ABC, ABD are two triangles on the same base AB and on the same side of it, the vertex of each triangle being outside the other. If $AC=AD$, shew that BC cannot $=BD$.

12. From C any point in a straight line AB, CD is drawn at right angles to AB, meeting a circle described with centre A and distance AB in D; and from AD, AE is cut off $=AC$: shew that AEB is a right angle.

Proposition XIII. Theorem.

The angles which one straight line makes with another upon one side of it are either two right angles, or together equal to two right angles.

Fig. 1.

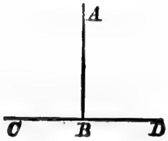
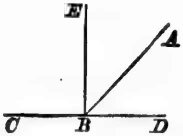


Fig. 2.



Let *AB* make with *CD* upon one side of it the ∠s *ABC*, *ABD*.

Then must these be either two rt. ∠s,
or together equal to two rt. ∠s

First, if ∠ *ABC*= ∠ *ABD* as in Fig. 1,
each of them is a rt. ∠. Def. 9.

Secondly, if ∠ *ABC* be not= ∠ *ABD*, as in Fig. 2,
from *B* draw *BE* ⊥ to *CD*. I. 11.

Then sum of ∠s *ABC*, *ABD*=sum of ∠s *EBC*, *EBA*, *ABD*,
and sum of ∠s *EBC*, *EBD*=sum of ∠s *EBC*, *EBA*, *ABD*;
∴ sum of ∠s *ABC*, *ABD*=sum of ∠s *EBC*, *EBD*;
Ax. 1.

∴ sum of ∠s *ABC*, *ABD*=sum of a rt. ∠ and a rt. ∠;
∴ ∠s *ABC*, *ABD* are together=two rt. ∠s.

Q. E. D.

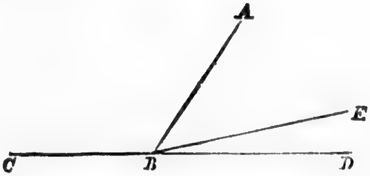
Ex. Straight lines drawn connecting the opposite angular points of a quadrilateral figure intersect each other in *O*. Shew that the angles at *O* are together equal to four right angles.

Note (1.) If two angles together make up a right angle, each is called the Complement of the other. Thus, in fig. 2. ∠ *ABD* is the complement of ∠ *ABE*.

(2.) If two angles together make up two right angles, each is called the Supplement of the other. Thus, in both figures, ∠ *ABD* is the supplement of ∠ *ABC*.

Proposition XIV. Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.



At the pt. *B* in the st. line *AB* let the st. lines *BC*, *BD*, on opposite sides of *AB*, make ∠ s *ABC*, *ABD* together = two rt. angles.

Then BD must be in the same st. line with BC.

For if not, let *BE* be in the same st. line with *BC*.

Then ∠ s *ABC*, *ABE* together = two rt. ∠ s. I. 13.

And ∠ s *ABC*, *ABD* together = two rt. ∠ s. Hyp.

∴ sum of ∠ s *ABC*, *ABE* = sum of ∠ s *ABC*, *ABD*.

Take away from each of these equals the ∠ *ABC* ;

then ∠ *ABE* = ∠ *ABD*, Ax. 3.

that is, the less = the greater ; which is impossible,

∴ *BE* is not in the same st. line with *BC*.

Similarly it may be shewn that no other line but *BD* is in the same st. line with *BC*.

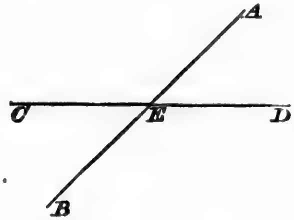
∴ *BD* is in the same st. line with *BC*.

Q. E. D.

Ex. Shew the necessity of the words *the opposite sides* in the enunciation.

Proposition XV. Theorem.

If two straight lines cut one another, the vertically opposite angles must be equal.



Let the st. lines AB, CD cut one another in the pt. E.

Then must $\angle\ AEC = \angle\ BED$ *and* $\angle\ AED = \angle\ BEC$.

For $\because$ AE meets CD,

$\therefore$ sum of $\angle$ s AEC, AED=two rt. $\angle$ s. I. 13.

And $\because$ DE meets AB,

$\therefore$ sum of $\angle$ s BED, AED=two rt. $\angle$ s; I. 13.

$\therefore$ sum of $\angle$ s AEC, AED=sum of $\angle$ s BED, AED;

$\therefore$ $\angle\ AEC = \angle\ BED$. Ax. 3.

Similarly it may be shewn that $\angle\ AED = \angle\ BEC$.

Q. E. D.

Corollary I. From this it is manifest, that if two straight lines cut one another, the four angles, which they make at the point of intersection, are together equal to four right angles.

Corollary II. All the angles, made by any number of straight lines meeting in one point, are together equal to four right angles.

Ex. 1. Shew that the bisectors of AED and BEC are in the same straight line.

Ex. 2. Prove that $\angle\ AED$ is equal to the angle between two straight lines drawn at right angles from E to AE and EC, if both lie above CD.

Ex. 3. If AB, CD bisect each other in E; shew that the triangles AED, BEC are equal in all respects.

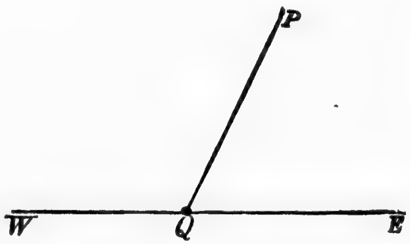
NOTE 3. *On Euclid's definition of an Angle.*

Euclid directs us to regard an angle as the inclination of two straight lines to each other, which meet, *but are not in the same straight line.*

Thus he does not recognise the existence of a single angle equal in magnitude to two right angles.

The words printed in italics are omitted as needless, in Def. VIII., p. 3, and that definition may be extended with advantage in the following terms :—

DEF. Let WQE be a fixed straight line, and QP a line which revolves about the fixed point Q, and which at first coincides with QE.



Then, when QP has reached the position represented in the diagram, we say that it has described the angle EQP.

When QP has revolved so far as to coincide with QW, we say that it has described an angle *equal to two right angles.*

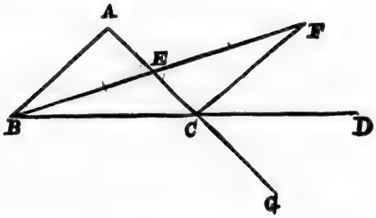
Hence we may obtain an easy proof of Prop. XIII. ; for whatever the position of PQ may be, the angles which it makes with WE are together equal to two right angles.

Again, in Prop. XV. it is evident that $\angle AED = \angle BEC$, since each has the same supplementary $\angle AEC$.

We shall shew hereafter, p. 149, how this definition may be extended, so as to embrace angles *greater than two right angles.*

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.



Let the side BC of $\triangle ABC$ be produced to D.

Then must $\angle ACD$ be greater than either $\angle CAB$ or $\angle ABC$.

Bisect AC in E, and join BE. I. 10.

Produce BE to F, making $EF=BE$, and join FC.

Then in $\triangle$s BEA, FEC,

$\because BE=FE$, and $EA=EC$, and $\angle BEA = \angle FEC$, I. 15.

$\therefore \angle ECF = \angle EAB$. I. 4.

Now $\angle ACD$ is greater than $\angle ECF$; Ax. 9.

$\therefore \angle ACD$ is greater than $\angle EAB$,

that is, $\angle ACD$ is greater than $\angle CAB$.

Similarly, if AC be produced to G it may be shewn that

$\angle BCG$ is greater than $\angle ABC$.

and $\angle BCG = \angle ACD$; I. 15.

$\therefore \angle ACD$ is greater than $\angle ABC$.

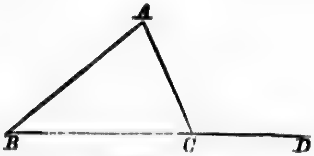
Q. E. D.

Ex. 1. From the same point there cannot be drawn more than two equal straight lines to meet a given straight line.

Ex. 2. If, from any point, a straight line be drawn to a given straight line making with it an acute and an obtuse angle, and if, from the same point, a perpendicular be drawn to the given line; the perpendicular will fall on the side of the acute angle.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.



Let ABC be any $\triangle$.

Then must any two of its $\angle$s be together less than two rt. $\angle$s.

Produce BC to D.

Then $\angle ACD$ is greater than $\angle ABC$. I. 16.

$\therefore$ $\angle$s ACD, ACB are together greater than $\angle$s ABC, ACB.

But $\angle$s ACD, ACB together = two rt. $\angle$s. I. 13

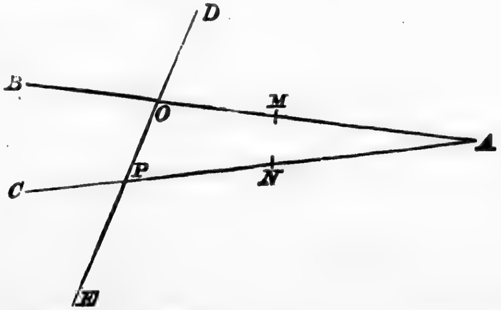
$\therefore$ $\angle$s ABC, ACB are together less than two rt. $\angle$s.

Similarly it may be shewn that $\angle$s ABC, BAC and also that $\angle$s BAC, ACB are together less than two rt. $\angle$s.

Q. E. D.

NOTE 4. *On the Sixth Postulate.*

We learn from Prop. XVII. that if two straight lines BM and CN, which meet in A, are met by another straight line DE in the points O, P,



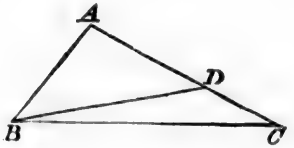
the angles MOP and NPO are together less than two right angles.

The Sixth Postulate asserts that if a line DE meeting two other lines BM, CN makes MOP, NPO, the two interior

angles on the same side of it, together less than two right angles, BM and CN shall meet if produced on the same side of DE on which are the angles MOP and NPO.

Proposition XVIII. Theorem.

If one side of a triangle be greater than a second, the angle opposite the first must be greater than that opposite the second.



In $\triangle ABC$, let side AC be greater than AB.

Then must $\angle ABC$ be greater than $\angle ACB$.

From AC cut off $AD = AB$, and join BD. I. 3.

Then $\because AB = AD$,

$\therefore \angle ADB = \angle ABD$, I. A.

And $\because CD$, a side of $\triangle BDC$, is produced to A.

$\therefore \angle ADB$ is greater than $\angle ACB$; I. 16

$\therefore$ also $\angle ABD$ is greater than $\angle ACB$.

Much more is $\angle ABC$ greater than $\angle ACB$.

Q. E. D.

Ex. Shew that if two angles of a triangle be equal, the sides which subtend them are equal also (Eucl. I. 6).

Proposition XIX. Theorem.

If one angle of a triangle be greater than a second, the side opposite the first must be greater than that opposite the second.

In $\triangle ABC$, let $\angle ABC$ be greater than $\angle ACB$.

Then must AC be greater than AB.

For if AC be not greater than AB,

AC must either $= AB$, or be less than AB.

Now AC cannot $= AB$, for then I. A.

$\angle ABC$ would $= \angle ACB$, which is not the case.

And AC cannot be less than AB, for then I. 18.

$\angle ABC$ would be less than $\angle ACB$, which is not the case;

$\therefore$ AC is greater than AB.

Q. E. D.

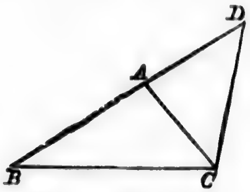
Ex. 1. In an obtuse-angled triangle, the greatest side is opposite the obtuse angle.

Ex. 2. BC, the base of an isosceles triangle BAC, is produced to any point D; shew that AD is greater than AB.

Ex. 3. The perpendicular is the shortest straight line, which can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than one more remote.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.



Let ABC be a $\triangle$.

Then any two of its sides must be together greater than the third side.

Produce BA to D, making $AD=AC$, and join DC.

Then $\because AD=AC$,

$\therefore \angle ACD = \angle ADC$, that is, $\angle BDC$. I. A.

Now $\angle BCD$ is greater than $\angle ACD$;

$\therefore \angle BCD$ is also greater than $\angle BDC$;

$\therefore BD$ is greater than BC. I. 19.

But $BD=BA$ and AD together;

that is, $BD=BA$ and AC together;

$\therefore BA$ and AC together are greater than BC.

Similarly it may be shewn that

AB and BC together are greater than AC,

and BC and CA .... AB.

Q. E. D.

Ex. 1. Prove that any three sides of a quadrilateral figure are together greater than the fourth side.

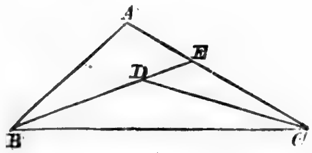
Ex. 2. Shew that any side of a triangle is greater than the difference between the other two sides.

Ex. 3. Prove that the sum of the distances of any point from the angular points of a quadrilateral is greater than half the perimeter of the quadrilateral.

Ex. 4. If one side of a triangle be bisected, the sum of the two other sides shall be more than double of the line joining the vertex and the point of bisection.

Proposition XXI. Theorem.

If, from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle; these will be together less than the other sides of the triangle, but will contain a greater angle.



Let *ABC* be a △, and from *D*, a pt. in the △, draw st. lines to *B* and *C*.

Then will BD, DC together be less than BA, AC, but ∠ BDC will be greater than ∠ BAC.

Produce *BD* to meet *AC* in *E*.

Then *BA, AE* are together greater than *BE*. I. 20.

Add to each *EC*.

Then *BA, AC* are together greater than *BE, EC*.

Again, *DE, EC* are together greater than *DC*. I. 20.

Add to each *BD*.

Then *BE, EC* are together greater than *BD, DC*.

And it has been shewn that *BA, AC* are together greater than *BE, EC*;

∴ *BA, AC* are together greater than *BD, DC*.

Next, ∵ ∠ *BDC* is greater than ∠ *DEC*, I. 16.

and ∠ *DEC* is greater than ∠ *BAC*, I. 16.

∴ ∠ *BDC* is greater than ∠ *BAC*.

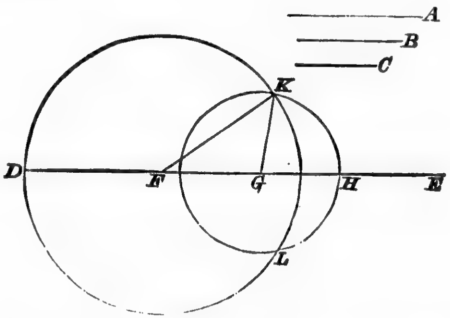
Q. E. D.

Ex. 1. Upon the base *AB* of a triangle *ABC* is described a quadrilateral figure *ADEB*, which is entirely within the triangle. Shew that the sides *AC, CB* of the triangle are together greater than the sides *AD, DE, EB* of the quadrilateral.

Ex. 2. Shew that the sum of the straight lines, joining the angles of a triangle with a point within the triangle, is less than the perimeter of the triangle, and greater than half the perimeter.

PROPOSITION XXII. PROBLEM.

To make a triangle, of which the sides shall be equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given lines, any two of which are together greater than the third.

It is required to make a $\triangle$ *having its sides* $= A$, B, C *respectively.*

Take a st. line DE of unlimited length.

In DE make $DF=A$, $FG=B$, and $GH=C$. I. 3.

With centre F and distance FD, describe $\odot DKL$.

With centre G and distance GH, describe $\odot HKL$.

Join FK and GK.

Then $\triangle KFG$ has its sides $=A$, B, C respectively.

For $FK=FD$; Def. 13.

$\therefore FK=A$;

and $GK=GH$; Def. 13.

$\therefore GK=C$;

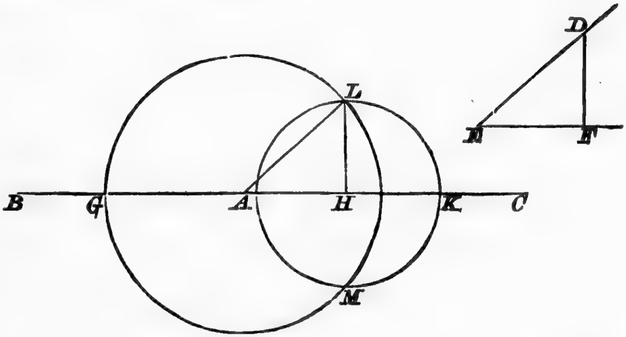
and $FG=B$;

$\therefore$ a $\triangle KFG$ has been described as reqd. Q. E. F.

Ex. Draw an isosceles triangle having each of the equal sides double of the base.

Proposition XXIII. Problem.

At a given point in a given straight line, to make an angle equal to a given angle.



Let A be the given pt., BC the given line, DEF the given $\angle$.

It is reqd. to make at pt. A an angle $= \angle DEF$.

In ED, EF take any pts. D. F; and join DF.

In AB, produced if necessary, make $AG = DE$.

In AC, produced if necessary, make $AH = EF$.

In HC, produced if necessary, make $HK = FD$.

With centre A, and distance AG, describe $\odot GLM$.

With centre H, and distance HK, describe $\odot LKM$.

Join AL and HL.

Then $\because LA = AG$, $\therefore LA = DE$; Ax. 1.

and $\because HL = HK$, $\therefore HL = FD$. Ax. 1.

Then in $\triangle$s LAH, DEF,

$\because LA = DE$, and $AH = EF$, and $HL = FD$;

$\therefore \angle LAH = \angle DEF$. I. c.

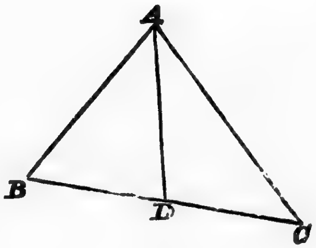
$\therefore$ an angle LAH has been made at pt. A as was reqd.

Q. E. F.

NOTE.—We here give the proof of a theorem, necessary to the proof of Prop. XXIV. and applicable to several propositions in Book III.

PROPOSITION D. THEOREM.

Every straight line, drawn from the vertex of a triangle to the base, is less than the greater of the two sides, or than either, if they be equal.



In the $\triangle ABC$, let the side AC be not less than AB.

Take any pt. D in BC, and join AD.

Then must AD be less than AC.

For $\because AC$ is not less than AB;

$\therefore \angle ABD$ is not less than $\angle ACD$. I. A. and 18.

But $\angle ADC$ is greater than $\angle ABD$; I. 16.

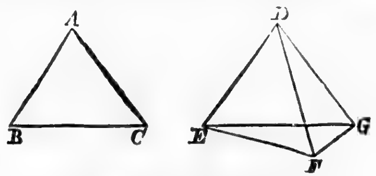
$\therefore \angle ADC$ is greater than $\angle ACD$;

$\therefore AC$ is greater than AD. I. 19.

Q. E. D.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle must be greater than the base of the other.



In the $\triangle$s ABC, DEF,
let $AB=DE$ and $AC=DF$,
and let $\angle BAC$ be greater than $\angle EDF$.
Then must BC be greater than EF.

Of the two sides DE, DF let DE be not greater than DF.*

At pt. D in st. line ED make $\angle EDG = \angle BAC$, I. 23.
and make $DG=AC$ or DF, and join EG, GF.

Then $\because AB=DE$, and $AC=DG$, and $\angle BAC = \angle EDG$,
$\therefore BC=EG$, I. 4.

Again, $\because DG=DF$,
$\therefore \angle DFG = \angle DGF$; I. A.
$\therefore \angle EFG$ is greater than $\angle DGF$;
much more then $\angle EFG$ is greater than $\angle EGF$;
$\therefore EG$ is greater than EF. I. 19.

But $EG=BC$;
$\therefore BC$ is greater than EF.

Q. E. D.

* This line was added by Simson to obviate a defect in Euclid's proof. *Without* this condition, three distinct cases must be discussed. *With* the condition, we can prove that F must lie below EG.

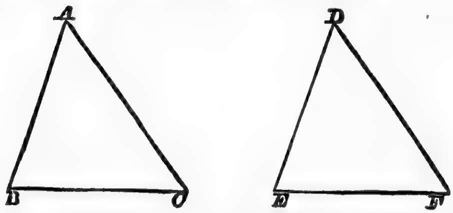
For since DF is not less than DE, and DG is drawn equal to DF, DG is not less than DE.

Hence by Prop. D, any line drawn from D to meet EG is less than DG, and therefore DF, being equal to DG, must extend beyond EG.

For another method of proving the Proposition, see p. 113.

Proposition XXV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also, contained by the sides of that which has the greater base, must be greater than the angle contained by the sides equal to them of the other.



In the $\triangle$s ABC, DEF,

let $AB = DE$ and $AC = DF$,

and let BC be greater than EF.

Then must $\angle BAC$ *be greater than* $\angle EDF$.

For $\angle BAC$ is greater than, equal to, or less than $\angle EDF$.

Now $\angle BAC$ cannot $= \angle EDF$,

for then, by I. 4, BC would $= EF$; which is not the case.

And $\angle BAC$ cannot be less than $\angle EDF$,

for then, by I. 24, BC would be less than EF; which is not the case;

$\therefore$ $\angle BAC$ must be greater than $\angle EDF$.

Q. E. D.

Note.—In Prop. XXVI. Euclid includes two cases, in which two triangles are equal in all respects; viz., when the following parts are equal in the two triangles:

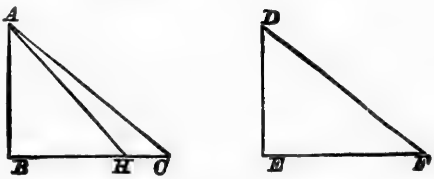
1. Two angles and the side between them.
2. Two angles and the side opposite one of them.

Of these we have already proved the first case, in Prop. B, so that we have only the second case left, to form the subject of Prop. XXVI., which we shall prove by the method of superposition.

For Euclid's proof of Prop. XXVI., see pp. 114-115.

PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, those sides being opposite to equal angles in each; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $AB = DE$.

Then must $BC = EF$, and $AC = DF$, and $\angle BAC = \angle EDF$.

Suppose $\triangle DEF$ to be applied to $\triangle ABC$,

so that D coincides with A, and DE falls on AB.

Then $\because DE = AB$, $\therefore E$ will coincide with B;

and $\because \angle DEF = \angle ABC$, $\therefore EF$ will fall on BC.

Then must F coincide with C: for, if not,

let F fall *between* B and C, at the pt. H. Join AH.

Then $\because \angle AHB = \angle DFE$, I. 4.

$\therefore \angle AHB = \angle ACB$,

the extr. $\angle$ = the intr. and opposite $\angle$, which is impossible.

$\therefore F$ does not fall between B and C.

Similarly, it may be shewn that F does not fall on BC produced.

$\therefore F$ coincides with C, and $\therefore BC = EF$;

$\therefore AC = DF$, and $\angle BAC = \angle EDF$, I. 4

and $\therefore$ the triangles are equal in all respects.

Q. E. D.

Miscellaneous Exercises on Props. I. to XXVI.

1. M is the middle point of the base BC of an isosceles triangle ABC, and N is a point in AC. Shew that the difference between MB and MN is less than that between AB and AN.

2. ABC is a triangle, and the angle at A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

3. AB, AC are straight lines meeting in A, and D is a given point. Draw through D a straight line cutting off equal parts from AB, AC.

4. Draw a straight line through a given point, to make equal angles with two given straight lines which meet.

5. A given angle BAC is bisected; if CA be produced to G and the angle BAG bisected, the two bisecting lines are at right angles.

6. Two straight lines are drawn to the base of a triangle from the vertex, one bisecting the vertical angle, and the other bisecting the base. Prove that the latter is the greater of the two lines.

7. Shew that Prop. XVII. may be proved without producing a side of the triangle.

8. Shew that Prop. XVIII. may be proved by means of the following construction: cut off $AD=AB$, draw AE, bisecting $\angle BAC$ and meeting BC in E, and join DE.

9. Shew that Prop. XX. can be proved, without producing one of the sides of the triangle, by bisecting one of the angles.

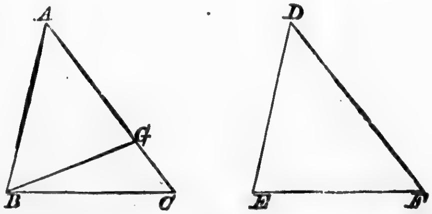
10. Given two angles of a triangle and the side adjacent to them, construct the triangle.

11. Shew that the perpendiculars, let fall on two sides of a triangle from any point in the straight line bisecting the angle contained by the two sides, are equal.

We conclude Section I. with the proof (omitted by Euclid) of another case in which two triangles are equal in all respects.

PROPOSITION E. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about a second angle in each equal: then, if the third angles in each be both acute, both obtuse, or if one of them be a right angle, the triangles are equal in all respects.



In the $\triangle$s ABC, DEF, let $\angle BAC = \angle EDF$, $AB = DE$, $BC = EF$, and let $\angle$s ACB, DFE be both acute, both obtuse, or let one of them be a right angle.

Then must $\triangle$s ABC, DEF be equal in all respects.

For if AC be not $= DF$, make $AG = DF$; and join BG.

Then in $\triangle$s BAG, EDF,

$\because BA = ED$, and $AG = DF$, and $\angle BAG = \angle EDF$,

$\therefore BG = EF$ and $\angle AGB = \angle DFE$. I. 4.

But $BC = EF$, and $\therefore BG = BC$;

$\therefore \angle BCG = \angle BGC$. I. A.

First, let $\angle ACB$ and $\angle DFE$ be both acute,

then $\angle AGB$ is acute, and $\therefore \angle BGC$ is obtuse; I. 13.

$\therefore \angle BCG$ is obtuse, which is contrary to the hypothesis.

Next, let $\angle ACB$ and $\angle DFE$ be both obtuse,

then $\angle AGB$ is obtuse, and $\therefore \angle BGC$ is acute; I. 13.

$\therefore \angle BCG$ is acute, which is contrary to the hypothesis.

Lastly, let one of the third angles ACB, DFE be a right angle.

If $\angle ACB$ be a rt. $\angle$,

then $\angle BGC$ is also a rt. $\angle$; I. A.

$\therefore$ $\angle$s BCG, BGC together = two rt. $\angle$s, which is impossible. I. 17.

Again, if $\angle DFE$ be a rt. $\angle$,

then $\angle AGB$ is a rt. $\angle$, and $\therefore$ $\angle BGC$ is a rt. $\angle$. I. 13.

Hence $\angle BCG$ is also a rt. $\angle$.

$\therefore$ $\angle$s BCG, BGC together = two rt. $\angle$s, which is impossible. I. 17.

Hence AC is equal to DF,

and the $\triangle$s ABC, DEF are equal in all respects.

Q. E. D.

COR. From the first case of this proposition we deduce the following important theorem:

If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are equal in all respects.

NOTE. In the enunciation of Prop. E, if, instead of the words *if one of them be a right angle*, we put the words *both right angles*, this case of the proposition would be identical with I. 26.

SECTION II.

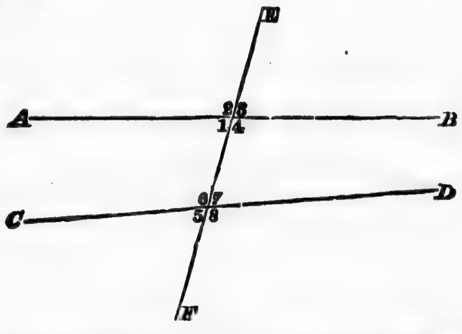
The Theory of Parallel Lines.

INTRODUCTION.

WE have detached the Propositions, in which Euclid treats of Parallel Lines, from those which precede and follow them in the First Book, in order that the student may have a clearer notion of the difficulties attending this division of the subject, and of the way in which Euclid proposes to meet them.

We must first explain some technical terms used in this Section.

If a straight line *EF* cut two other straight lines *AB*, *CD*, it makes with those lines eight angles, to which particular names are given.



The angles numbered 1, 4, 6, 7 are called *Interior* angles.
.... 2, 3, 5, 8 .... *Exterior* ....

The angles marked 1 and 7 are called *alternate* angles.

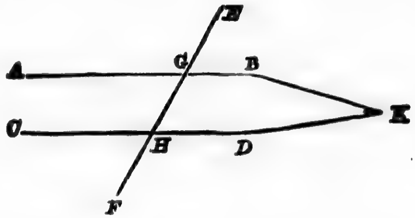
The angles marked 4 and 6 are also called alternate angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7 are called *corresponding* angles.

NOTE. From I. 13 it is clear that the angles 1, 4, 6, 7 are together equal to four right angles.

Proposition XXVII. Theorem.

If a straight line, falling upon two other straight lines, make the alternate angles equal to one another; these two straight lines must be parallel.



Let the st. line EF, falling on the st. lines AB, CD, make the alternate $\angle$s AGH, GHD equal.

Then must AB be $\parallel$ to CD.

For if not, AB and CD will meet, if produced, either towards B, D, or towards A, C.

Let them be produced and meet towards B, D in K.

Then GHK is a $\triangle$;

and $\therefore$ $\angle AGH$ is greater than $\angle GHD$. I. 16.

But $\angle AGH = \angle GHD$, Hyp.

which is impossible.

$\therefore$ AB, CD do not meet when produced towards B, D.

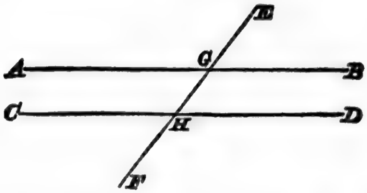
In like manner it may be shewn that they do not meet when produced towards A, C.

$\therefore$ AB and CD are parallel. Def. 26.

Q. E. D.

Proposition XXVIII. Theorem.

If a straight line, falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles; the two straight lines are parallel to one another.



Let the st. line EF, falling on st. lines AB, CD, make

I. $\angle\, EGB$ = corresponding $\angle\, GHD$, or

II. $\angle$ s BGH, GHD together = two rt. $\angle$ s.

Then, in either case, AB must be $\parallel$ to CD.

I.	$\because \angle\, EGB$ is given $= \angle\, GHD$,	Hyp.
	and $\angle\, EGB$ is known to be $= \angle\, AGH$,	I. 15.
	$\therefore \angle\, AGH = \angle\, GHD$;	
	and these are alternate $\angle$ s;	
	$\therefore AB$ is $\parallel$ to CD.	I. 27.
II.	$\because \angle$ s BGH, GHD together = two rt. $\angle$ s,	Hyp.
	and $\angle$ s BGH, AGH together = two rt. $\angle$ s,	I. 13.
	$\therefore \angle$ s BGH, AGH together $= \angle$ s BGH, GHD together;	
	$\therefore \angle\, AGH = \angle\, GHD$;	
	$\therefore AB$ is $\parallel$ to CD.	I. 27.

Q. E. D.

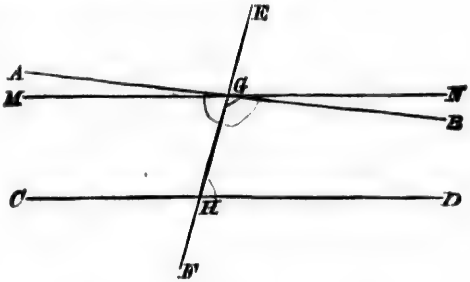
NOTE 5. *On the Sixth Postulate.*

In the place of Euclid's Sixth Postulate many modern writers on Geometry propose, as more evident to the senses, the following Postulate :—

"*Two straight lines which cut one another cannot* BOTH *be parallel to the same straight line.*"

If this be assumed, we can prove Post. 6, as a Theorem, thus :

Let the line *EF* falling on the lines *AB*, *CD* make the ∠ s *BGH*, *GHD* together less than two rt. ∠ s. Then must *AB*, *CD* meet when produced towards *B*, *D*.



For if not, suppose *AB* and *CD* to be parallel.

Then ∵ ∠ s *AGH*, *BGH* together = two rt. ∠ s, I. 13.

and ∠ s *GHD*, *BGH* are together less than two rt. ∠ s,

∴ ∠ *AGH* is greater than ∠ *GHD*.

Make ∠ *MGH* = ∠ *GHD*, and produce *MG* to *N*.

Then ∵ the alternate ∠ s *MGH*, *GHD* are equal,

∴ *MN* is ∥ to *CD*. I. 27.

Thus two lines *MN*, *AB* which cut one another are both parallel to *CD*, which is impossible.

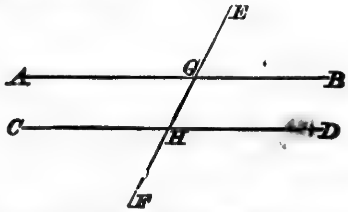
∴ *AB* and *CD* are not parallel.

It is also clear that they meet towards *B*, *D*, because *GB* lies between *GN* and *HD*.

Q. E. D

Proposition XXIX. Theorem.

If a straight line fall upon two parallel straight lines, it makes the two interior angles upon the same side together equal to two right angles, and also the alternate angles equal to one another, and also the exterior angle equal to the interior and opposite upon the same side.



Let the st. line *EF* fall on the parallel st. lines *AB*, *CD*.

Then must

I. ∠s *BGH*, *GHD* together = two rt. ∠s.

II. ∠ *AGH* = alternate ∠ *GHD*.

III. ∠ *EGB* = corresponding ∠ *GHD*.

I. ∠s *BGH*, *GHD* cannot be together *less* than two rt. ∠s, for then *AB* and *CD* would meet if produced towards *B* and *D*, Post. 6.
which cannot be, for they are parallel.

Nor can ∠s *BGH*, *GHD* be together *greater* than two rt. ∠s,
for then ∠s *AGH*, *GHC* would be together less than two rt. ∠s, I. 13.
and *AB*, *CD* would meet if produced towards *A* and *C* Post. 6
which cannot be, for they are parallel,
∴ ∠s *BGH*, *GHD* together = two rt. ∠s.

II. ∵ ∠s *BGH*, *GHD* together = two rt. ∠s,
and ∠s *BGH*, *AGH* together = two rt. ∠s, I. 13.
∴ ∠s *BGH*, *AGH* together = ∠s *BGH*, *GHD* together,
and ∴ ∠ *AGH* = ∠ *GHD*. Ax. 3.

III. ∵ ∠ *AGH* = ∠ *GHD*,
and ∠ *AGH* = ∠ *EGB*, I. 15.
∴ ∠ *EGB* = ∠ *GHD*. Ax. 1

Q. E. D.

EXERCISES.

1. If through a point, equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines; they will intercept equal portions of the parallel lines.

2. If a straight line be drawn, bisecting one of the angles of a triangle, to meet the opposite side; the straight lines drawn from the point of section, parallel to the other sides and terminated by those sides, will be equal.

3. If any straight line joining two parallel straight lines be bisected, any other straight line, drawn through the point of bisection to meet the two lines, will be bisected in that point.

NOTE. One Theorem (A) is said to be the *converse* of another Theorem (B), when the hypothesis in (A) is the conclusion in (B), and the conclusion in (A) is the hypothesis in (B).

For example, the Theorem I. A. may be stated thus:

Hypothesis. If two sides of a triangle be equal.

Conclusion. The angles opposite those sides must also be equal.

The converse of this is the Theorem I. B. Cor.:

Hypothesis. If two angles of a triangle be equal.

Conclusion. The sides opposite those angles must also be equal.

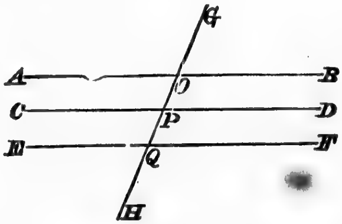
The following are other instances;

Postulate VI. is the converse of I. 17.

I. 29 is the converse of I. 27 and 28.

Proposition XXX. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the st. lines AB, CD be each $\parallel$ to EF.

Then must AB be $\parallel$ to CD.

Draw the st. line GH, cutting AB, CD, EF in the pts. O, P, Q.

Then $\because GH$ cuts the $\parallel$ lines AB, EF,

$\therefore \angle AOP =$ alternate $\angle PQF$. I. 29.

And $\because GH$ cuts the $\parallel$ lines CD, EF,

$\therefore$ extr. $\angle OPD =$ intr. $\angle PQF$; I. 29.

$\therefore \angle AOP = \angle OPD$;

and these are alternate angles;

$\therefore AB$ is $\parallel$ to CD. I. 27.

Q. E. D.

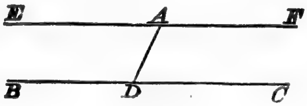
The following Theorems are important. They admit of easy proof, and are therefore left as Exercises for the student.

1. If two straight lines be parallel to two other straight lines, each to each, the first pair make the same angles with one another as the second.

2. If two straight lines be perpendicular to two other straight lines, each to each, the first pair make the same angles with one another as the second.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given pt. and BC the given st. line.

It is required to draw through A a st. line $\parallel$ to BC.

In BC take any pt. D, and join AD.

Make $\angle DAE = \angle ADC$. I. 23.

Produce EA to F. Then EF shall be $\parallel$ to BC.

For $\because$ AD, meeting EF and BC, makes the alternate angles equal, that is, $\angle EAD = \angle ADC$,

$\therefore$ EF is $\parallel$ to BC. I. 27.

$\therefore$ a st. line has been drawn through A $\parallel$ to BC.

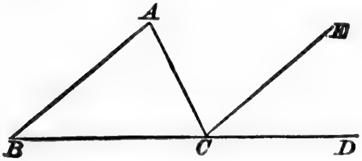
Q. E. F.

Ex. 1. From a given point draw a straight line, to make an angle with a given straight line that shall be equal to a given angle.

Ex. 2. Through a given point A draw a straight line ABC, meeting two parallel straight lines in B and C, so that BC may be equal to a given straight line.

Proposition XXXII. Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.



Let *ABC* be a △, and let one of its sides, *BC*, be produced to *D*.

Then will

I. ∠ *ACD* = ∠ s *ABC*, *BAC together.*

II. ∠ s *ABC*, *BAC*, *ACB together = two rt.* ∠ *s.*

From *C* draw *CE* ∥ to *AB*. I. 31.

Then I. ∵ *BD* meets the ∥s *EC*, *AB*,

∴ extr. ∠ *ECD* = intr. ∠ *ABC*. I. 29.

And ∵ *AC* meets the ∥s *EC*, *AB*,

∴ ∠ *ACE* = alternate ∠ *BAC*. I. 29.

∴ ∠ s *ECD*, *ACE* together = ∠ s *ABC*, *BAC* together ;

∴ ∠ *ACD* = ∠ s *ABC*, *BAC* together.

And II. ∵ ∠ s *ABC*, *BAC* together = ∠ *ACD*,

to each of these equals add ∠ *ACB* ;

then ∠ s *ABC*, *BAC*, *ACB* together = ∠ s *ACD*, *ACB* together,

∴ ∠ s *ABC*, *BAC*, *ACB* together = two rt. ∠ s. I. 13.

Q. E. D.

Ex. 1. In an acute-angled triangle, any two angles are greater than the third.

Ex. 2. The straight line, which bisects the external vertical angle of an isosceles triangle is parallel to the base.

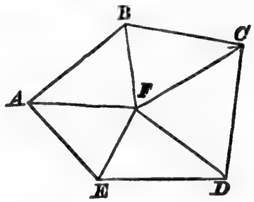
Ex. 3. If the side BC of the triangle ABC be produced to D, and AE be drawn bisecting the angle BAC and meeting BC in E; shew that the angles ABD, ACD are together double of the angle AED.

Ex. 4. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet; shew that they will contain an angle equal to an exterior angle at the base of the triangle.

Ex. 5. If the straight line bisecting the external angle of a triangle be parallel to the base; prove that the triangle is isosceles.

The following Corollaries to Prop. 32 were first given in Simson's Edition of Euclid.

Cor. 1. *The sum of the interior angles of any rectilinear figure together with four right angles is equal to twice as many right angles as the figure has sides.*



Let $ABCDE$ be any rectilinear figure.

Take any pt. F within the figure, and from F draw the st. lines FA, FB, FC, FD, FE to the angular pts. of the figure

Then there are formed as many $\angle$s as the figure has sides.

The three $\angle$s in *each* of these $\triangle$s together = two rt. $\angle$s.

$\therefore$ *all* the $\angle$s in these $\triangle$s together = twice as many right $\angle$s as there are $\triangle$s, that is, twice as many right $\angle$s as the figure has sides.

Now angles of all the $\triangle$s = $\angle$s at A, B, C, D, E and $\angle$s at F,

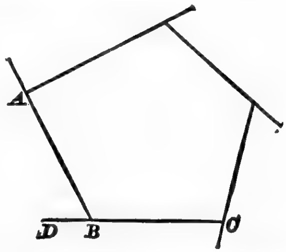
that is, = $\angle$s of the figure and $\angle$s at F,

and $\therefore$ = $\angle$s of the figure and four rt. $\angle$s. I. 15. Cor. 2

$\therefore$ $\angle$s of the figure and four rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides.

Cor. 2. *The exterior angles of any convex rectilinear figure, made by producing each of its sides in succession, are together equal to four right angles.*

Every interior angle, as *ABC*, and its adjacent exterior angle, as *ABD*, together are = two rt. ∠ s.



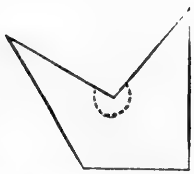
∴ all the intr. ∠ s together with all the extr. ∠ s
= twice as many rt. ∠ s as the figure has sides.

But all the intr. ∠ s together with four rt. ∠ s
= twice as many rt. ∠ s as the figure has sides.

∴ all the intr. ∠ s together with all the extr. ∠ s
= all the intr. ∠ s together with four rt. ∠ s.

∴ all the extr. ∠ s = four rt. ∠ s.

Note. The latter of these corollaries refers only to *convex* figures, that is, figures in which every interior angle is less than two right angles. When a figure contains an angle greater



than two right angles, as the angle marked by the dotted line in the diagram, this is called a reflex *angle*. See p. 149.

Ex. 1. The exterior angles of a quadrilateral made by producing the sides successively are together equal to the interior angles.

Ex. 2. Prove that the interior angles of a hexagon are equal to eight right angles.

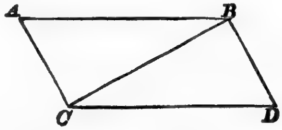
Ex. 3. Shew that the angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

Ex. 4. How many sides has the rectilinear figure, the sum of whose interior angles is double that of its exterior angles?

Ex. 5. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.



Let the equal and ∥ st. lines AB, CD be joined towards the same parts by the st. lines AC, BD.

Then must AC and BD be equal and ∥.

Join BC.

Then $\because AB$ is ∥ to CD,

$\therefore \angle ABC =$ alternate $\angle DCB$. I. 29.

Then in $\triangle$s ABC, BCD,

$\because AB = CD$, and BC is common, and $\angle ABC = \angle DCB$,

$\therefore AC = BD$, and $\angle ACB = \angle DBC$. I. 4.

Then $\because BC$, meeting AC and BD,

makes the alternate $\angle$s ACB, DBC equal,

$\therefore AC$ is ∥ to BD.

Q. E. D.

Miscellaneous Exercises on Sections I. and II.

1. If two exterior angles of a triangle be bisected by straight lines which meet in O; prove that the perpendiculars from O on the sides, or the sides produced, of the triangle are equal.

2. Trisect a right angle.

3. The bisectors of the three angles of a triangle meet in one point.

4. The perpendiculars to the three sides of a triangle drawn from the middle points of the sides meet in one point.

5. The angle between the bisector of the angle BAC of the triangle ABC and the perpendicular from A on BC, is equal to half the difference between the angles at B and C.

6. If the straight line AD bisect the angle at A of the triangle ABC, and BDE be drawn perpendicular to AD, and meeting AC, or AC produced, in E; shew that BD is equal to DE.

7. Divide a right-angled triangle into two isosceles triangles.

8. AB, CD are two given straight lines. Through a point E between them draw a straight line GEH, such that the intercepted portion GH shall be bisected in E.

9. The vertical angle O of a triangle OPQ is a right, acute, or obtuse angle, according as OR, the line bisecting PQ, is equal to, greater or less than the half of PQ.

10. Shew by means of Ex. 9 how to draw a perpendicular to a given straight line from its extremity without producing it.

SECTION III.

On the Equality of Rectilinear Figures in respect of Area.

THE amount of space enclosed by a Figure is called the Area of that figure.

Euclid calls two figures *equal* when they enclose the same amount of space. They may be dissimilar in shape, but if the areas contained within the boundaries of the figures be the same, then he calls the figures *equal*. He regards a triangle, for example, as a figure having sides and angles and area, and he proves in this section that two triangles may have equality of area, though the sides and angles of each may be unequal.

Coincidence of their boundaries is a test of the equality of all geometrical magnitudes, as we explained in Note 1, page 14.

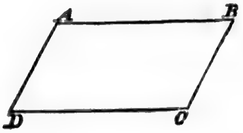
In the case of lines and angles it is the only test: in the case of *figures* it is *a test, but not the only test;* as we shall shew in this Section.

The sign =, standing between the symbols denoting two *figures*, must be read *is equal in area to.*

Before we proceed to prove the Propositions included in this Section, we must complete the list of Definitions required in Book I., continuing the numbers prefixed to the definitions in page 6.

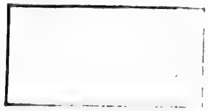
DEFINITIONS.

XXVII. A PARALLELOGRAM is a four-sided figure whose opposite sides are parallel.



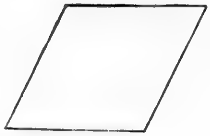
For brevity we often designate a parallelogram by two letters only, which mark opposite angles. Thus we call the figure in the margin the parallelogram *AC*.

XXVIII. A Rectangle is a parallelogram, having one of its angles a right angle.

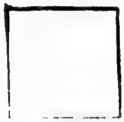


Hence by I. 29, *all* the angles of a rectangle are right angles.

XXIX. A RHOMBUS is a parallelogram, having its sides equal.



XXX. A SQUARE is a parallelogram, having its sides equal and one of its angles a right angle.



Hence, by I. 29, *all* the angles of a square are right angles.

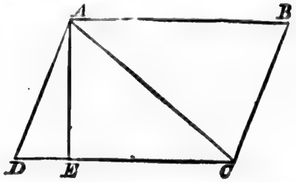
XXXI. A TRAPEZIUM is a four-sided figure of which two sides only are parallel.

XXXII. A DIAGONAL of a four-sided figure is the straight line joining two of the opposite angular points.

XXXIII. The ALTITUDE of a Parallelogram is the perpendicular distance of one of its sides from the side opposite, regarded as the Base.

The altitude of a triangle is the perpendicular distance of one of its angular points from the side opposite, regarded as the base.

Thus if $ABCD$ be a parallelogram, and AE a perpendicular let fall from A to CD, AE is the *altitude* of the parallelogram, and also of the triangle ACD.



If a perpendicular be let fall from B to DC produced, meeting DC in F, BF is the altitude of the parallelogram.

EXERCISES.

Prove the following theorems:

1. The diagonals of a square make with each of the sides an angle equal to half a right angle.

2. If two straight lines bisect each other, the lines joining their extremities will form a parallelogram.

3. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

4. If the straight lines joining two opposite angular points of a parallelogram bisect the angles, the parallelogram has all its sides equal.

5. If the opposite angles of a quadrilateral be equal, the quadrilateral is a parallelogram.

6. If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.

7. If one angle of a rhombus be equal to two-thirds of two right angles, the diagonal drawn from that angular point divides the rhombus into two equilateral triangles.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it.

Let $ABDC$ be a ▱, and BC a diagonal of the ▱.

Then must $AB = DC$ and $AC = DB$,

and $\angle BAC = \angle CDB$, and $\angle ABD = \angle ACD$

and $\triangle ABC = \triangle DCB$.

For $\because AB$ is ∥ to CD, and BC meets them,

$\therefore \angle ABC =$ alternate $\angle DCB$; I. 29

and $\because AC$ is ∥ to BD, and BC meets them,

$\therefore \angle ACB =$ alternate $\angle DBC$. I. 29.

Then in $\triangle$s ABC, DCB,

$\because \angle ABC = \angle DCB$, and $\angle ACB = \angle DBC$,

and BC is common, a side adjacent to the equal $\angle$s in each;

$\therefore AB = DC$, and $AC = DB$, and $\angle BAC = \angle CDB$,

and $\triangle ABC = \triangle DCB$. I. B.

Also $\because \angle ABC = \angle DCB$, and $\angle DBC = \angle ACB$,

$\therefore \angle$s ABC, DBC together $= \angle$s DCB, ACB together,

that is, $\angle ABD = \angle ACD$.

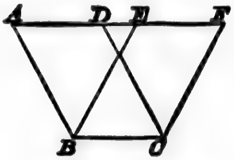
Q. E. D.

Ex. 1. Shew that the diagonals of a parallelogram bisect each other.

Ex. 2. Shew that the diagonals of a rectangle are equal.

PROPOSITION XXXV. THEOREM.

Parallelograms on the same base and between the same parallels are equal.



Let the ▱s $ABCD$, $EBCF$ be on the same base BC and between the same ||s AF, BC.

Then must ▱ $ABCD =$ ▱ $EBCF$.

CASE I. If AD, EF have no point common to both,

Then in the △s FDC, EAB,

∵ extr. $\angle FDC =$ intr. $\angle EAB$,	I. 29.
and intr. $\angle DFC =$ extr. $\angle AEB$,	I. 29.
and $DC = AB$,	I. 34.
∴ $\triangle FDC = \triangle EAB$.	I. 26.

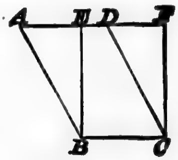
Now ▱ $ABCD$ with △ $FDC =$ figure $ABCF$;

and ▱ $EBCF$ with △ $EAB =$ figure $ABCF$;

∴ ▱ $ABCD$ with △ $FDC =$ ▱ $EBCF$ with △ EAB;

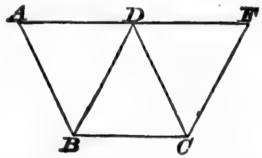
∴ ▱ $ABCD =$ ▱ $EBCF$.

CASE II. If the sides AD, EF overlap one another



the same method of proof applies.

Case III. If the sides opposite to BC be terminated in the same point D,



the same method of proof is applicable,
but it is easier to reason thus:

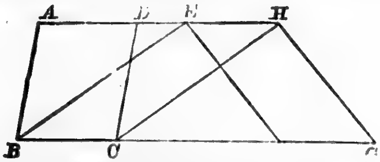
Each of the ▱s is double of △ BDC; I. 34.

$\therefore$ ▱ $ABCD$ = ▱ $DBCF$.

Q. E. D.

Proposition XXXVI. Theorem.

Parallelograms on equal bases, and between the same parallels, are equal to one another.



Let the ▱s $ABCD$, $EFGH$ be on equal bases BC, FG, and between the same $\parallel$s AH, BG.

Then must ▱ $ABCD$ = ▱ $EFGH$.

Join BE, CH.

Then $\because BC = FG$, Hyp.

and $EH = FG$; I. 34.

$\therefore BC = EH$;

and BC is $\parallel$ to EH. Hyp.

$\therefore EB$ is $\parallel$ to CH; I. 33.

$\therefore EBCH$ is a parallelogram.

Now ▱ $EBCH$ = ▱ $ABCD$, I. 35.

$\because$ they are on the same base BC and between the same $\parallel$s;

and ▱ $EBCH$ = ▱ $EFGH$, I. 35.

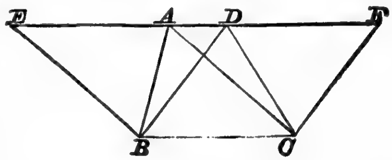
$\because$ they are on the same base EH and between the same $\parallel$s,

$\therefore$ ▱ $ABCD$ = ▱ $EFGH$.

Q. E. D.

Proposition XXXVII. Theorem.

Triangles upon the same base, and between the same parallels, are equal to one another.



Let $\triangle$s ABC, DBC be on the same base BC and between the same $\parallel$s AD, BC.

Then must $\triangle\ ABC = \triangle\ DBC$.

From B draw $BE \parallel$ to CA to meet DA produced in E.

From C draw $CF \parallel$ to BD to meet AD produced in F.

Then $EBCA$ and $FCBD$ are parallelograms,

and $\square\ EBCA = \square\ FCBD$, I. 35.

$\because$ they are on the same base and between the same $\parallel$s.

Now $\triangle\ ABC$ is half of $\square\ EBCA$, I. 34.

and $\triangle\ DBC$ is half of $\square\ FCBD$; I. 34.

$\therefore \triangle\ ABC = \triangle\ DBC$. Ax. 7.

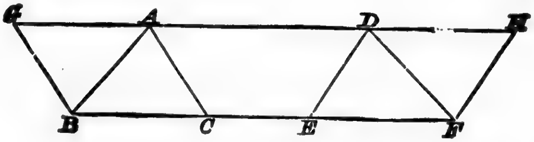
Q. E. D.

Ex. 1. If P be a point in a side AB of a parallelogram $ABCD$, and PC, PD be joined, the triangles PAD, PBC are together equal to the triangle PDC.

Ex. 2. If A, B be points in one, and C, D points in another of two parallel straight lines, and the lines AD, BC intersect in E, then the triangles AEC, BED are equal.

Proposition XXXVIII. Theorem.

Triangles upon equal bases, and between the same parallels, are equal to one another.



Let $\triangle$s ABC, DEF be on equal bases, BC, EF, and between the same $\parallel$s BF, AD.

Then must $\triangle\ ABC = \triangle\ DEF$.

From B draw $BG \parallel$ to CA to meet DA produced in G.

From F draw $FH \parallel$ to ED to meet AD produced in H.

Then CG and EH are parallelograms, and they are equal,

$\because$ they are on equal bases BC, EF, and between the same $\parallel$s BF, GH. I. 36

Now $\triangle\ ABC$ is half of $\square\ CG$,

and $\triangle\ DEF$ is half of $\square\ EH$;

$\therefore\ \triangle\ ABC = \triangle\ DEF$. Ax. 7.

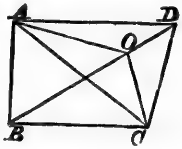
Q. E. D.

Ex. 1. Shew that a straight line, drawn from the vertex of a triangle to bisect the base, divides the triangle into two equal parts.

Ex. 2. In the equal sides AB, AC of an isosceles triangle ABC points D, E are taken such that $BD = AE$. Shew that the triangles CBD, ABE are equal.

Proposition XXXIX. Theorem.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.



Let the equal $\triangle$s ABC, DBC be on the same base BC, and on the same side of it.

Join AD.

Then must AD be $\parallel$ to BC.

For if not, through A draw $AO \parallel$ to BC, so as to meet BD, or BD produced, in O, and join OC.

Then $\because$ $\triangle$s ABC, OBC are on the same base and between the same $\parallel$s,

$\therefore \triangle ABC = \triangle OBC.$ I. 37.

But $\triangle ABC = \triangle DBC$; Hyp.

$\therefore \triangle OBC = \triangle DBC,$

the less = the greater, which is impossible;

$\therefore AO$ is not $\parallel$ to BC.

In the same way it may be shewn that no other line passing through A but AD is $\parallel$ to BC;

$\therefore AD$ is $\parallel$ to BC.

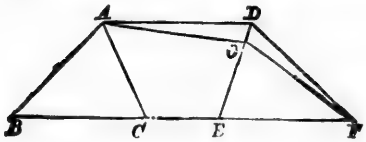
Q. E. D.

Ex. 1. AD is parallel to BC; AC, BD meet in E; BC is produced to P so that the triangle PEB is equal to the triangle ABC: shew that PD is parallel to AC.

Ex. 2. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

PROPOSITION XL. THEOREM.

Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.



Let the equal $\triangle$s ABC, DEF be on equal bases BC, EF in the same st. line BF and towards the same parts.

Join AD.

Then must AD be $\parallel$ to BF.

For if not, through A draw $AO \parallel$ to BF, so as to meet ED, or ED produced, in O, and join OF.

Then $\triangle ABC = \triangle OEF$, $\because$ they are on equal bases and between the same $\parallel$s. I. 38.

But $\triangle ABC = \triangle DEF$; Hyp.

$\therefore \triangle OEF = \triangle DEF$,

the less = the greater, which is impossible.

$\therefore AO$ is not $\parallel$ to BF.

In the same way it may be shewn that no other line passing through A but AD is $\parallel$ to BF,

$\therefore AD$ is $\parallel$ to BF.

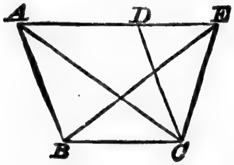
Q. E. D.

Ex. 1. The straight line, joining the points of bisection of two sides of a triangle, is parallel to the base, and is equal to half the base.

Ex. 2. The straight lines, joining the middle points of the sides of a triangle, divide it into four equal triangles.

Proposition XLI. Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.



Let the ▱ *ABCD* and the △ *EBC* be on the same base *BC* and between the same ||s *AE*, *BC*.

Then must ▱ *ABCD be double of* △ *EBC.*

Join *AC*.

Then △ *ABC* = △ *EBC*, ∵ they are on the same base and between the same ||s ; I. 37.

and ▱ *ABCD* is double of △ *ABC*, ∵ *AC* is a diagonal of *ABCD* ; I. 34.

∴ ▱ *ABCD* is double ot △ *EBC*.

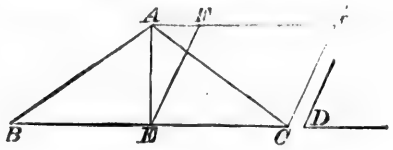
Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given $\triangle$, and D the given $\angle$.

It is required to describe a $\square$ equal to $\triangle$ ABC, having one of its $\angle$s $= \angle$ D.

Bisect BC in E and join AE. I. 10.

At E make $\angle$ $CEF = \angle$ D. I. 23.

Draw AFG $\parallel$ to BC, and from C draw CG $\parallel$ to EF.

Then $FECG$ is a parallelogram.

Now $\triangle$ $AEB = \triangle$ AEC,

$\because$ they are on equal bases and between the same $\parallel$s. I. 38.

$\therefore$ $\triangle$ ABC is double of $\triangle$ AEC.

But $\square$ $FECG$ is double of $\triangle$ AEC,

$\because$ they are on same base and between same $\parallel$s. I. 41.

$\therefore$ $\square$ $FECG = \triangle$ ABC; Ax. 6.

and $\square$ $FECG$ has one of its $\angle$s, $CEF = \angle$ D.

$\therefore$ $\square$ $FECG$ has been described as was reqd.

Q. E. F.

Ex. 1. Describe a triangle, which shall be equal to a given parallelogram, and have one of its angles equal to a given rectilineal angle.

again

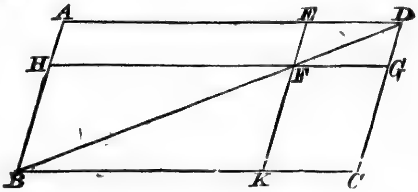
Ex. 2. Construct a parallelogram, equal to a given triangle, and such that the sum of its sides shall be equal to the sum of the sides of the triangle.

Ex. 3. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equal to, the given triangle.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Diagonal



Let $ABCD$ be a ▱, of which BD is a diagonal, and EG, HK the ▱s about BD, that is, through which BD passes,

and AF, FC the other ▱s, which make up the whole figure $ABCD$,

and which are ∴ called the Complements.

Then must complement AF = complement FC.

For ∵ BD is a diagonal of ▱ AC,

∴ $\triangle ABD = \triangle CDB$; I. 34.

and ∵ BF is a diagonal of ▱ HK,

∴ $\triangle HBF = \triangle KFB$; I. 34.

and ∵ FD is a diagonal of ▱ EG,

∴ $\triangle EFD = \triangle GDF$. I. 34.

Hence sum of $\triangle$s HBF, EFD = sum of $\triangle$s KFB, GDF.

Take these equals from $\triangle$s ABD, CDB respectively,

then remaining ▱ AF = remaining ▱ FC. Ax. 3.

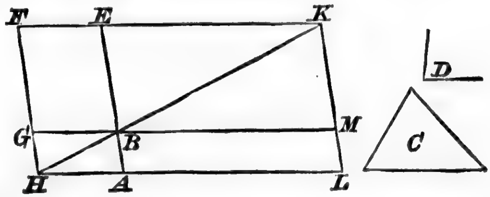
Q. E. D.

Ex. 1. If through a point O, within a parallelogram $ABCD$, two straight lines are drawn parallel to the sides, and the parallelograms OB, OD are equal; the point O is in the diagonal AC.

Ex. 2. $ABCD$ is a parallelogram, AMN a straight line meeting the sides BC, CD (one of them being produced) in M, N. Shew that the triangle MBN is equal to the triangle MDC.

Proposition XLIV. Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let AB be the given st. line, C the given △, D the given ∠.

It is required to apply to AB a ▱ = △ C and having one of its ∠s = ∠ D.

Make a ▱ = △ C, and having one of its angles = ∠ D, I. 42.

and suppose it to be removed to such a position that one of the sides containing this angle is in the same st. line with AB, and let the ▱ be denoted by $BEFG$.

Produce FG to H, draw AH ∥ to BG or EF, and join BH.

Then ∵ FH meets the ∥s AH, EF,

∴ sum of ∠s AHF, HFE = two rt. ∠s; I. 29.

∴ sum of ∠s BHG, HFE is less than two rt. ∠s;

∴ HB, FE will meet if produced towards B, E. Post. 6.

Let them meet in K.

Through K draw KL ∥ to EA or FH,

and produce HA, GB to meet KL in the pts. L, M.

Then $HFKL$ is a ▱, and HK is its diagonal;

and AG, ME are ▱s about HK,

∴ complement BL = complement BF, I. 43

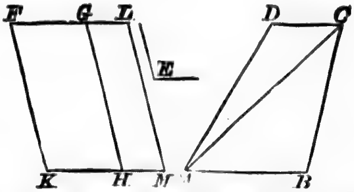
∴ ▱ BL = △ C.

Also the ▱ BL has one of its ∠s, ABM = ∠ EBG, and ∴ equal to ∠ D.

Q. E. F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram, which shall be equal to a given rectilinear figure, and have one of its angles equal to a given angle.



Let $ABCD$ be the given rectil. figure, and E the given $\angle$.

It is required to describe a $\square = $ *to* $ABCD$, *having one of its* $\angle$*s* $= \angle E$.

Join AC.

Describe a $\square$ $FGHK = \triangle ABC$, having $\angle FKH = \angle E$. I. 42.

To GH apply a $\square$ $GHML = \triangle CDA$, having $\angle GHM = \angle E$. I. 44.

Then $FKML$ is the $\square$ reqd.

For $\because$ $\angle GHM$ and $\angle FKH$ are each $= \angle E$;

$\therefore$ $\angle GHM = \angle FKH$,

$\therefore$ sum of $\angle$s GHM, GHK = sum of $\angle$s FKH, GHK

= two rt. $\angle$s; I. 29.

$\therefore$ KHM is a st. line. I. 14.

Again, $\because$ HG meets the $\|$s FG, KM,

$\angle FGH = \angle GHM$,

$\therefore$ sum of $\angle$s FGH, LGH = sum of $\angle$s GHM, LGH

= two rt. $\angle$s; I. 29.

$\therefore$ FGL is a st. line. I. 14.

Then $\because$ KF is $\|$ to HG, and HG is $\|$ to LM

$\therefore$ KF is $\|$ to LM; I. 30.

and KM has been shewn to be $\|$ to FL,

$\therefore$ $FKML$ is a parallelogram,

and $\because$ $FH = \triangle ABC$, and $GM = \triangle CDA$,

$\therefore$ $\square$ FM = whole rectil. fig. $ABCD$,

and $\square$ FM has one of its $\angle$s, $FKM = \angle E$.

In the same way a $\square$ may be constructed equal to a given rectil. fig. of any number of sides, and having one of its angles equal to a given angle. Q. E. F.

Miscellaneous Exercises.

1. If one diagonal of a quadrilateral bisect the other, it divides the quadrilateral into two equal triangles.

2. If from any point in the diagonal, or the diagonal produced, of a parallelogram, straight lines be drawn to the opposite angles, they will cut off equal triangles.

3. In a trapezium the straight line, joining the middle points of the parallel sides, bisects the trapezium.

4. The diagonals *AC*, *BD* of a parallelogram intersect in *O*, and *P* is a point within the triangle *AOB*; prove that the difference of the triangles *CPD*, *APD* is equal to the sum of the triangles *APC*, *BPD*.

5. If either diagonal of a parallelogram be equal to a side of the figure, the other diagonal shall be greater than any side of the figure.

6. If through the angles of a parallelogram four straight lines be drawn parallel to its diagonals, another parallelogram will be formed, the area of which will be double that of the original parallelogram.

7. If two triangles have two sides respectively equal and the included angles supplemental, the triangles are equal.

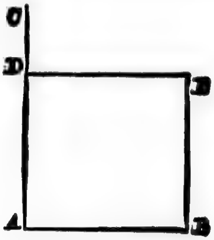
8. Bisect a given triangle by a straight line drawn from a given point in one of the sides.

9. The base *AB* of a triangle *ABC* is produced to a point *D* such that *BD* is equal to *AB*, and straight lines are drawn from *A* and *D* to *E*, the middle point of *BC*; prove that the triangle *ADE* is equal to the triangle *ABC*.

10. Prove that a pair of the diagonals of the parallelograms, which are about the diameter of any parallelogram, are parallel to each other.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.



Let AB be the given st. line.

It is required to describe a square on AB.

From A draw $AC \perp$ to AB. I. 11. Cor.

In AC make $AD=AB$.

Through D draw $DE \parallel$ to AB. I. 31.

Through B draw $BE \parallel$ to AD. I. 31.

Then AE is a parallelogram,

and $\therefore AB=ED$, and $AD=BE$. I. 34.

But $AB=AD$;

$\therefore AB, BE, ED, DA$ are all equal;

$\therefore AE$ is equilateral.

And $\angle BAD$ is a right angle.

$\therefore AE$ is a square, Def. XXX.

and it is described on AB.

Q. E. F.

Ex. 1. Shew how to construct a rectangle whose sides are equal to two given straight lines.

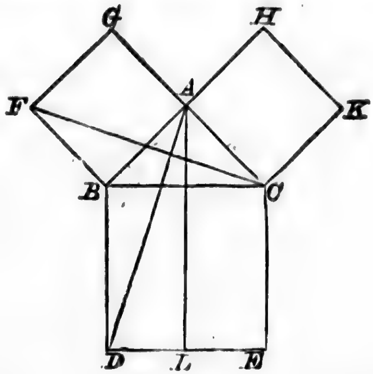
Ex. 2. Shew that the squares on equal straight lines are equal.

Ex. 3. Shew that equal squares must be on equal straight lines.

NOTE. The theorems in Ex. 2 and 3 are assumed by Euclid in the proof of Prop. XLVIII.

Proposition XLVII. Theorem.

In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.



Let ABC be a right-angled $\triangle$, having the rt. $\angle BAC$.

Then must sq. on BC = sum of sqq. on BA, AC.

On BC, CA, AB descr. the sqq. $BDEC$, $CKHA$, $AGFB$.

Through A draw $AL \parallel$ to BD or CE, and join AD, FC.

Then $\because \angle BAC$ and $\angle BAG$ are both rt. $\angle$s,

$\therefore CAG$ is a st. line; I. 14

and $\because \angle BAC$ and $\angle CAH$ are both rt. $\angle$s;

$\therefore BAH$ is a st. line. I. 14.

Now $\because \angle DBC = \angle FBA$, each being a rt. $\angle$,

adding to each $\angle ABC$, we have

$\angle ABD = \angle FBC$. Ax. 2.

Then in $\triangle$s ABD, FBC,

$\because AB = FB$, and $BD = BC$, and $\angle ABD = \angle FBC$,

$\therefore \triangle ABD = \triangle FBC$. I. 4.

Now $\square$ BL is double of $\triangle ABD$, on same base BD and between same $\parallel$s AL, BD; I. 41.

and sq. BG is double of $\triangle FBC$, on same base FB and between same $\parallel$s FB, GC; I. 41.

$\therefore \square$ BL = sq. BG.

Similarly, by joining AE, BK it may be shewn that

$$\square\, CL = \text{sq. } AK.$$

Now sq. on BC = sum of $\square\, BL$ and $\square\, CL$,

= sum of sq. BG and sq. AK,

= sum of sqq. on BA and AC.

Q. E. D.

Ex. 1. Prove that the square, described upon the diagonal of any given square, is equal to twice the given square.

Ex. 2. Find a line, the square on which shall be equal to the sum of the squares on three given straight lines.

Ex. 3. If one angle of a triangle be equal to the sum of the other two, and one of the sides containing this angle being divided into four equal parts, the other contains three of those parts; the remaining side of the triangle contains five such parts.

Ex. 4. The triangles ABC, DEF, having the angles ACB, DFE right angles, have also the sides AB, AC equal to DE, DF, each to each; shew that the triangles are equal in every respect.

Note. This Theorem has been already deduced as a Corollary from Prop. E, page 43.

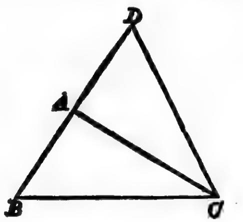
Ex. 5. Divide a given straight line into two parts, so that the square on one part shall be double of the square on the other.

Ex. 6. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and on the line so drawn are together equal to the sum of the squares on the segment adjacent to the right angle and on the hypotenuse.

Ex. 7. In any triangle, if a line be drawn from the vertex at right angles to the base, the difference between the squares on the sides is equal to the difference between the squares on the segments of the base.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by those sides is a right angle.



Let the sq. on BC, a side of $\triangle ABC$, be equal to the sum of the sqq. on AB, AC.

Then must $\angle BAC$ *be a rt. angle.*

From pt. A draw $AD \perp$ to AC. I. 11.

Make $AD=AB$, and join DC.

Then $\because AD=AB$,

$\therefore$ sq. on AD=sq. on AB; I. 46, Ex. 2.

add to each sq. on AC.

then sum of sqq. on AD, AC=sum of sqq. on AB, AC.

But $\because \angle DAC$ is a rt. angle,

$\therefore$ sq. on DC=sum of sqq. on AD, AC; I. 47.

and, by hypothesis,

sq. on BC=sum of sqq. on AB, AC;

$\therefore$ sq. on DC=sq. on BC;

$\therefore DC=BC$. I. 46, Ex. 3.

Then in $\triangle$s ABC, ADC,

$\because AB=AD$, and AC is common, and $BC=DC$,

$\therefore \angle BAC = \angle DAC$; I. c.

and $\angle DAC$ is a rt. angle, by construction;

$\therefore \angle BAC$ is a rt. angle.

Q. E. D.

BOOK II.

INTRODUCTORY REMARKS.

THE geometrical figure with which we are chiefly concerned in this book is the RECTANGLE. A rectangle is said to be *contained by* any two of its adjacent sides.

Thus if $ABCD$ be a rectangle, it is said to be contained by AB, AD, or by any other pair of adjacent sides.

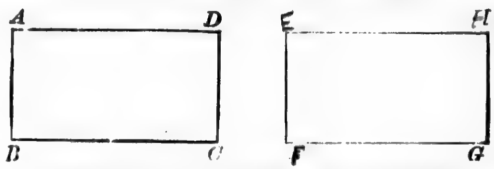
We shall use the abbreviation *rect.* AB, AD to express the words "the rectangle contained by AB, AD."

We shall make frequent use of a Theorem (employed, but not demonstrated, by Euclid) which may be thus stated and proved.

PROPOSITION A. THEOREM.

If the adjacent sides of one rectangle be equal to the adjacent sides of another rectangle, each to each, the rectangles are equal in area.

Let $ABCD$, $EFGH$ be two rectangles:
and let $AB=EF$ and $BC=FG$.



Then must rect. $ABCD$ = *rect.* $EFGH$.

For if the rect. $EFGH$ be applied to the rect. $ABCD$, so that EF coincides with AB,

then FG will fall on BC, $\because \angle EFG = \angle ABC$,

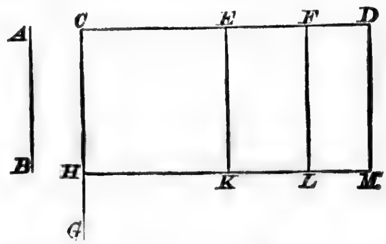
and G will coincide with C, $\because BC=FG$.

Similarly it may be shewn that H will coincide with D,

$\therefore$ rect. $EFGH$ coincides with and is therefore equal to rect $ABCD$. Q. E. D.

Proposition I. Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and CD be two given st. lines,
and let CD be divided into any parts in E, F.

Then must rect. AB, CD = *sum of rect.* AB, CE *and rect.* AB, EF *and rect.* AB, FD.

From C draw $CG \perp$ to CD, and in CG make $CH = AB$.
Through H draw $HM \parallel$ to CD. I. 31.
Through E, F, and D draw EK, FL, $DM \parallel$ to CH.
Then EK and FL, being each $= CH$, are each $= AB$.

Now CM = sum of CK and EL and FM.

And CM = rect. AB, CD, $\because CH = AB$,
CK = rect. AB, CE, $\because CH = AB$,
EL = rect. AB, EF, $\because EK = AB$,
FM = rect. AB, FD, $\because FL = AB$;

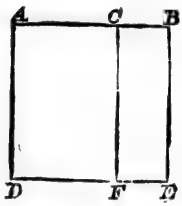
$\therefore$ rect. AB, CD = sum of rect. AB, CE and rect. AB, EF and rect. AB, FD.

Q. E. D.

Ex. If two straight lines be each divided into any number of parts, the rectangle contained by the two lines is equal to the rectangles contained by all the parts of the one taken separately with all the parts of the other.

Proposition II. Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.



Let the st. line AB be divided into any two parts in C.

Then must

sq. on AB = sum of rect. AB, AC and rect. AB, CB.

On AB describe the sq. $ADEB$ I. 46.

Through C draw $CF \parallel$ to AD.   I. 31.

Then AE = sum of AF and CE.

Now AE is the sq. on AB,

AF = rect. AB, AC, $\because AD = AB$,

CE = rect. AB, CB, $\because BE = AB$,

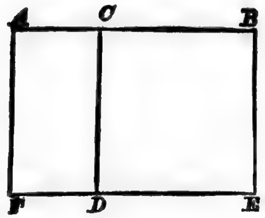
$\therefore$ sq. on AB = sum of rect. AB, AC and rect. AB, CB.

Q. E. D.

Ex. The square on a straight line is equal to four times the square on half the line.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.



Let the st. line AB be divided into any two parts in C.

Then must

rect. AB, CB=sum of rect. AC, CB and sq. on CB.

On CB describe the sq. $CDEB$. I. 46

From A draw $AF \parallel$ to CD, meeting ED produced in F.

Then AE=sum of AD and CE.

Now AE=rect. AB, CB, $\because BE=CB$,

AD=rect. AC, CB, $\because CD=CB$,

CE=sq. on CB.

$\therefore$ rect. AB, CB=sum of rect. AC, CB and sq. on CB.

Q. E. D.

NOTE. When a straight line is cut in a point, the distances of the point of section from the ends of the line are called the *segments* of the line.

If a line AB be divided in C,

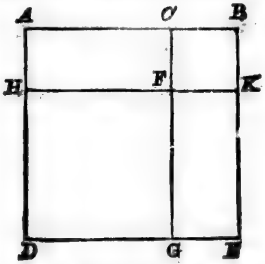
AC and CB are called the *internal* segments of AB.

If a line AC be produced to B,

AB and CB are called the *external* segments of AC.

Proposition IV. Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts together with twice the rectangle contained by the parts.



Let the st. line AB be divided into any two parts in C.

Then must

sq. on AB=sum of sqq. on AC, CB and twice rect. AC, CB.

On AB describe the sq. $ADEB$. I. 46.

From AD cut off $AH=CB$. Then $HD=AC$. Ax 3

Draw $CG \parallel$ to AD, and $HK \parallel$ to AB, meeting CG in F. I 31

Then $\because BK=AH$, 34 $\therefore BK=CB$, Ax. I.

$\therefore BK, KF, FC, CB$ are all equal; and KBC is a rt. $\angle$;

$\therefore CK$ is the sq. on CB. Def. XXX.

Also HG=sq. on AC, $\because HF$ and HD each$=AC$.

Now AE=sum of HG, CK, AF, FE,

and AE=sq. on AB,

HG=sq. on AC,

CK=sq. on CB,

AF=rect. AC, CB, $\because CF=CB$,

FE=rect. AC, CB, $\because FG=AC$ and $FK=CB$.

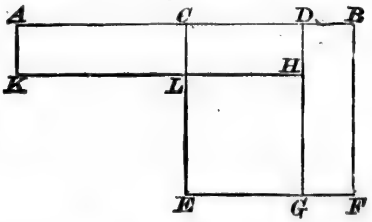
$\therefore$ sq. on AB=sum of sqq. on AC, CB and twice rect. AC, CB.

Q. E. D.

Ex. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base. Shew that the rectangle, contained by the segments of the base, is equal to the square on the perpendicular.

Proposition V. Theorem.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.



Let the st. line AB be divided equally in C and unequally in D.

Then must

rect. AD, DB together with sq. on CD=sq. on CB.

On CB describe the sq. $CEFB$. I. 46.

Draw $DG \parallel$ to CE, and from it cut off $DH=DB$. I. 31.

Draw $HLK \parallel$ to AD, and $AK \parallel$ to DH. I. 31.

Then rect. DF=rect. AL, $\because BF=AC$, and $BD=CL$.

Also LG=sq. on CD, $\because LH=CD$, and $HG=CD$.

Then rect. AD, DB together with sq. on CD

$=AH$ together with LG

=sum of AL and CH and LG

=sum of DF and CH and LG

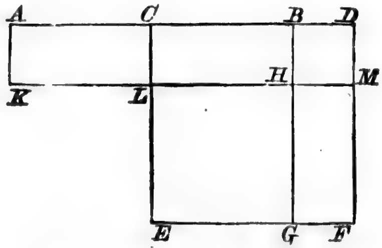
$=CF$

=sq. on CB.

Q. E. D.

PROPOSITION VI. THEOREM.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.



Let the st. line AB **be bisected in** C **and produced to** D.

Then must

rect. AD, DB *together with sq. on* CB = *sq. on* CD.

On CD describe the sq. $CEFD$. I. 46.

Draw $BG \parallel$ to CE, and cut off $BH = BD$. I. 31 I 3

Through H draw $KLM \parallel$ to AD I. 31.

Through A draw $AK \parallel$ to CE. I 31

Now $\because BG = CD$ and $BH = BD$;

$\therefore HG = CB$; Ax. 3.

$\therefore$ rect. MG = rect. AL. II. A.

Then rect. AD, DB together with sq. on CB

= sum of AM and LG

= sum of AL and CM and LG

= sum of MG and CM and LG

= CF,

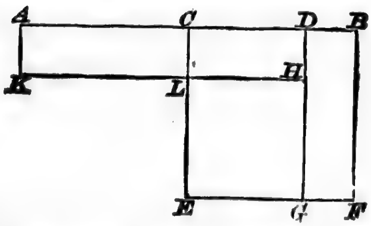
= sq. on CD.

Q. E. D.

NOTE. We here give the proof of an important theorem, which is usually placed as a corollary to Proposition V.

PROPOSITION B. THEOREM.

The difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.



Let AC, CD be two st. lines, of which AC is the greater, and let them be placed so as to form one st. line AD.

Produce AD to B, making $CB=AC$.

Then $AD=$the sum of the lines AC, CD,

and $DB=$the difference of the lines AC, CD.

Then must difference between sqq. on AC, $CD=$rect. AD, DB.

On CB describe the sq. $CEFB$. I. 46.

Draw $DG \parallel$ to CE, and from it cut off $DH=DB$. I. 31.

Draw $HLK \parallel$ to AD, and $AK \parallel$ to DH. I. 31.

Then rect. $DF=$rect. AL, $\because BF=AC$, and $BD=CL$.

Also $LG=$sq. on CD, $\because LH=CD$, and $HG=CD$.

Then difference between sqq. on AC, CD

=difference between sqq. on CB, CD

=sum of CH and DF

=sum of CH and AL

$=AH$

=rect. AD, DH

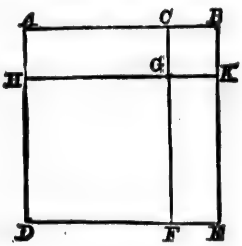
=rect. AD, DB.

Q. E. D.

Ex. Shew that Propositions V. and VI. might be deduced from this Proposition.

Proposition VII. Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part together with the square on the other part.



Let AB be divided into any two parts in C.

Then must

sqq. on AB, BC=twice rect. AB, BC together with sq. on AC.

On AB describe the sq. $ADEB$. I. 46.

From AD cut off $AH=CB$.

Draw $CF \parallel$ to AD and $HGK \parallel$ to AB. I. 31.

Then HF=sq. on AC, and CK=sq. on CB.

Then sqq. on AB, BC=sum of AE and CK

=sum of AK, HF, GE and CK

=sum of AK, HF and CE.

Now AK=rect. AB, BC, $\because BK=BC$;

CE=rect. AB, BC, $\because BE=AB$;

HF=sq. on AC.

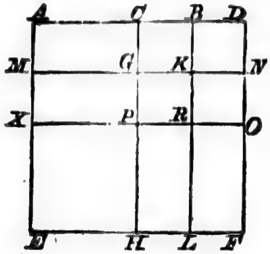
$\therefore$ sqq. on AB, BC=twice rect. AB, BC together with sq. on AC.

Q. E. D.

Ex. If straight lines be drawn from G to B and from G to D, shew that BGD is a straight line.

Proposition VIII. Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first part.



Let the st. line AB be divided into any two parts in C.

Produce AB to D, so that $BD = BC$.

Then must four times rect. AB, BC together with sq. on AC = sq. on AD.

On AD describe the sq. $AEFD$. I. 46.

From AE cut off AM and MX each $= CB$.

Through C, B draw CH, $BL \parallel$ to AE. I. 31.

Through M, X draw $MGKN$, $XPRO \parallel$ to AD. I. 31.

Now $\because XE = AC$, and $XP = AC$, $\therefore XH =$ sq. on AC.

Also $AG = MP = PL = RF$, II. A.

and $CK = GR = BN = KO$; II. A.

$\therefore$ sum of these eight rectangles

$=$ four times the sum of AG, CK

$=$ four times AK

$=$ four times rect. AB, BC.

Then four times rect. AB, BC and sq. on AC

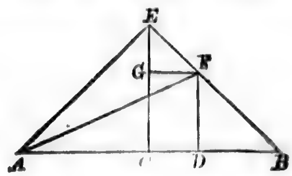
$=$ sum of the eight rectangles and XH

$= AEFD$

$=$ sq. on AD. Q. E. D.

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.



Let AB be divided equally in C and unequally in D.

Then must

sum of sqq. on AD, DB = twice sum of sqq. on AC, CD.

Draw $CE = AC$ at rt. ∠ s to AB, and join EA, EB.

Draw DF at rt. ∠ s to AB, meeting EB in F.

Draw FG at rt. ∠ s to EC, and join AF.

Then ∵ ∠ ACE is a rt. ∠,

∴ sum of ∠ s AEC, EAC = a rt. ∠; I. 32.

and ∵ ∠ AEC = ∠ EAC, I. A.

∴ ∠ AEC = half a rt. ∠.

So also ∠ BEC and ∠ EBC are each = half a rt. ∠.

Hence ∠ AEF is a rt. ∠.

Also, ∵ ∠ GEF is half a rt. ∠, and ∠ EGF is a rt. ∠;

∴ ∠ EFG is half a rt. ∠;

∴ ∠ EFG = ∠ GEF, and ∴ $EG = GF$. I. B. Cor.

So also ∠ BFD is half a rt. ∠, and $BD = DF$.

Now sum of sqq. on AD, DB

= sq. on AD together with sq. on DF

= sq. on AF I. 47.

= sq. on AE together with sq. on EF I. 47.

= sqq. on AC, EC together with sqq. on EG, GF I. 47.

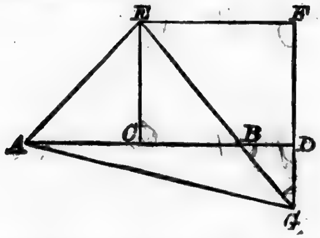
= twice sq. on AC together with twice sq. on GF

= twice sq. on AC together with twice sq. on CD.

Q. E. D.

Proposition X. Theorem.

If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.



Let the st. line AB be bisected in C and produced to D.

Then must

sum of sqq. on AD, BD = twice sum of sqq. on AC, CD.

Draw $CE \perp$ to AB, and make $CE = AC$.

Join EA, EB and draw $EF \parallel$ to AD and $DF \parallel$ to CE.

Then $\because$ $\angle$s FEB, EFD are together less than two rt. $\angle$s,

$\therefore$ EB and FD will meet if produced towards B, D in some pt. G.

Join AG.

Then $\because$ $\angle ACE$ is a rt. $\angle$, I (32)

$\therefore$ $\angle$s EAC, AEC together = a rt. $\angle$,

and $\because$ $\angle EAC = \angle AEC$, I. A.

$\therefore$ $\angle AEC$ = half a rt. $\angle$.

So also $\angle$s BEC, EBC each = half a rt. $\angle$.

$\therefore$ $\angle AEB$ is a rt. $\angle$.

Also $\angle DBG$, which = $\angle EBC$, is half a rt. $\angle$, I (15)

and $\therefore$ $\angle BGD$ is half a rt. $\angle$;

$\therefore$ $BD = DG$. I. B. Cor.

Again, $\because$ $\angle FGE$ = half a rt. $\angle$, and $\angle EFG$ is a rt. $\angle$, I. 34.

$\therefore$ $\angle FEG$ = half a rt. $\angle$, and $EF = FG$. I. B. Cor.

Then sum of sqq. on AD, DB

= sum of sqq. on AD, DG

= sq. on AG I. 47.

= sq. on AE together with sq. on EG I. 47.

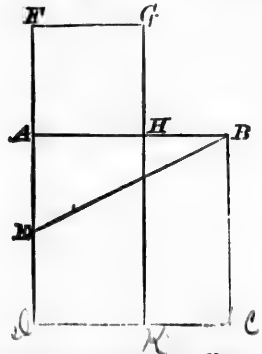
= sqq. on AC, EC together with sqq. on EF, FG I. 47.

= twice sq. on AC together with twice sq. on EF

= twice sq. on AC together with twice sq. on CD. Q. E. D.

PROPOSITION XI. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Let AB be the given st. line.

On AB descr. the sq. $ADCB$. I. 46.

Bisect AD in E and join EB. I. 10.

Produce DA to F, making $EF = EB$.

On AF descr. the sq. $AFGH$. I. 46.

Then AB is divided in H so that rect. $AB, BH =$ sq. on AH.

Produce GH to K.

Then $\because DA$ is bisected in E and produced to F,

$\therefore$ rect. DF, FA together with sq. on AE

$=$ sq. on EF II. 6.

$=$ sq. on EB, $\because EB = EF$,

$=$ sum of sqq. on AB, AE. I. 47.

Take from each the square on AE.

Then rect. $DF, FA =$ sq. on AB. Ax. 3.

Now $FK =$ rect. DF, FA, $\because FG = FA$.

$\therefore FK = AC$.

Take from each the common part AK.

Then $FH = HC$;

that is, sq. on $AH =$ rect. AB, BH, $\because BC = AB$.

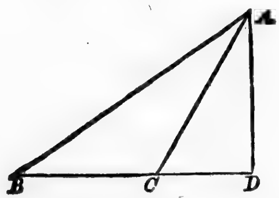
Thus AB is divided in H as was reqd.

Q. E. F.

Ex. Shew that the squares on the whole line and one of the parts are equal to three times the square on the other part.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side, upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.



Let ABC be an obtuse-angled $\triangle$, having $\angle ACB$ obtuse.

From A draw $AD \perp$ to BC produced.

Then must sq. on AB be greater than sum of sqq. on BC, CA by twice rect. BC, CD.

For since BD is divided into two parts in C,
sq. on BD = sum of sqq. on BC, CD, and twice rect. BC, CD. II. 4.

Add to each sq. on DA: then
sum of sqq. on BD, DA = sum of sqq. on BC, CD, DA and twice rect. BC, CD.

Now sqq. on BD, DA = sq. on AB, I. 47.
and sqq. on CD, DA = sq. on CA; I. 47.

$\therefore$ sq. on AB = sum of sqq. on BC, CA and twice rect. BC, CD.

$\therefore$ sq. on AB is greater than sum of sqq. on BC, CA by twice rect. BC, CD.

Q. E. D.

Ex. The squares on the diagonals of a trapezium are together equal to the squares on its two sides, which are not parallel, and twice the rectangle contained by the sides, which are parallel.

Proposition XIII. Theorem.

In every triangle, the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the acute angle.

Fig. 1.

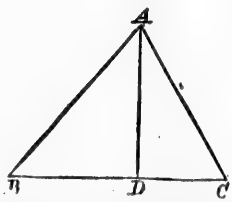
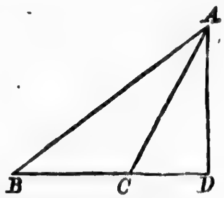


Fig. 2.



Let ABC be any $\triangle$, having the $\angle ABC$ acute.

From A draw $AD \perp$ to BC or BC produced.

Then must sq. on AC be less than the sum of sqq. on AB, BC, by twice rect. BC, BD.

For in Fig. 1 BC is divided into two parts in D,
and in Fig. 2 BD is divided into two parts in C;

$\therefore$ in both cases

sum of sqq. on BC, BD = sum of twice rect. BC, BD and sq. on CD. II. 7.

Add to each the sq. on DA, then

sum of sqq. on BC, BD, DA = sum of twice rect. BC, BD and sqq. on CD, DA;

$\therefore$ sum of sqq. on BC, AB = sum of twice rect. BC, BD and sq. on AC; I. 47.

$\therefore$ sq. on AC is less than sum of sqq. on AB, BC by twice rect. BC, BD.

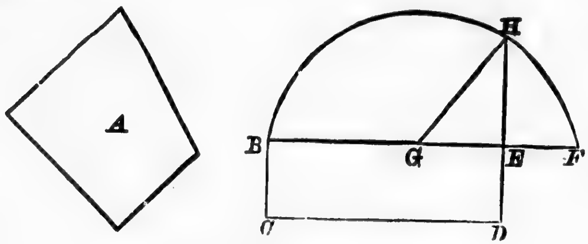
The case, in which the perpendicular AD coincides with AC, needs no proof.

Q. E. D.

Ex. Prove that the sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base and on the line joining the vertical angle with the middle point of the base.

Proposition XIV. Problem.

To describe a square that shall be equal to a given rectilinear figure.



Let A be the given rectil. figure.

It is reqd. to describe a square that shall $=A$.

Describe the rectangular ▱ $BCDE=A$. I. 45.

Then if $BE=ED$ the ▱ $BCDE$ is a square, and what was reqd. is done.

But if BE be not $=ED$, produce BE to F, so that $EF=ED$.

Bisect BF in G; and with centre G and distance GB, describe the semicircle BHF.

Produce DE to H and join GH.

Then, $\because$ BF is divided equally in G and unequally in E,

$\therefore$ rect. BE, EF together with sq. on GE

$=$sq. on GF II. 5.

$=$sq. on GH

$=$sum of sqq. on EH, GE. I. 47.

Take from each the square on GE.

Then rect. BE, $EF=$sq. on EH.

But rect. BE, $EF=BD$, $\because$ $EF=ED$;

$\therefore$ sq. on $EH=BD$;

$\therefore$ sq. on $EH=$rectil. figure A.

Q. E. F.

Miscellaneous Exercises on Book II.

1. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base; shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

2. The squares on the diagonals of a parallelogram are together equal to the squares on the four sides.

3. If $ABCD$ be any rectangle, and O any point either within or without the rectangle, shew that the sum of the squares on OA, OC is equal to the sum of the squares on OB, OD.

4. If either diagonal of a parallelogram be equal to one of the sides about the opposite angle of the figure, the square on it shall be less than the square on the other diameter, by twice the square on the other side about that opposite angle.

5. Produce a given straight line AB to C, so that the rectangle, contained by the sum and difference of AB and AC, may be equal to a given square.

6. Shew that the sum of the squares on the diagonals of any quadrilateral is less than the sum of the squares on the four sides, by four times the square on the line joining the middle points of the diagonals.

7. If the square on the perpendicular from the vertex of a triangle is equal to the rectangle, contained by the segments of the base, the vertical angle is a right angle.

8. If two straight lines be given, shew how to produce one of them so that the rectangle contained by it and the produced part may be equal to the square on the other.

9. If a straight line be divided into three parts, the square on the whole line is equal to the sum of the squares on the parts together with twice the rectangle contained by each two of the parts.

10. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. If straight lines be drawn from each angle of a triangle to bisect the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

12. CD is drawn perpendicular to AB, a side of the triangle ABC, in which $AC = AB$. Shew that the square on CD is equal to the square on BD together with twice the rectangle AD, DB.

13. The hypotenuse AB of a right-angled triangle ABC is trisected in the points D, E; prove that if CD, CE be joined, the sum of the squares on the sides of the triangle CDE is equal to two-thirds of the square on AB.

14. The square on the hypotenuse of an isosceles right angled triangle is equal to four times the square on the perpendicular from the right angle on the hypotenuse.

15. Divide a given straight line into two parts, so that the rectangle contained by them shall be equal to the square described upon a straight line, which is less than half the line divided.

Note 6.—*On the Measurement of Areas.*

To measure a Magnitude, we fix upon some magnitude of the same kind to serve as a standard or unit; and then any magnitude of that kind is measured by the number of times it contains this unit, and this number is called the Measure of the quantity.

Suppose, for instance, we wish to measure a straight line *AB*. We take another straight line *EF* for our standard,

and then we say

if *AB* contain *EF* three times, the measure of *AB* is 3,
if ....four...4,
if .... x ....x.

Next suppose we wish to measure two straight lines *AB*, *CD* by the same standard *EF*.

If *AB* contain *EF* m times
and *CD* .... n times,

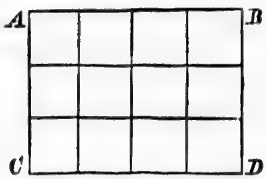
where m and n stand for numbers, whole or fractional, we say that *AB* and *CD* are *commensurable*.

But it may happen that we may be able to find a standard line *EF*, such that it is contained an exact number of times in *AB*; and yet there is no number, whole or fractional, which will express the number of times *EF* is contained in *CD*.

In such a case, where no unit-line can be found, such that it is contained an exact number of times in *each* of two lines *AB*, *CD*, these two lines are called *incommensurable*.

In the processes of Geometry we constantly meet with incommensurable magnitudes. Thus the side and diagonal of a square are incommensurables; and so are the diameter and circumference of a circle.

Next, suppose two lines AB, AC to be at right angles to each other and to be commensurable, so that AB contains four times a certain unit of linear measurement, which is contained by AC three times.



Divide AB, AC into four and three equal parts respectively, and draw lines through the points of division parallel to AC, AB respectively; then the rectangle $ACDB$ is divided into a number of equal squares, each constructed on a line equal to the unit of linear measurement.

If one of these squares be taken as the unit of area, the *measure* of the area of the rectangle $ACDB$ will be the number of these squares.

Now this number will evidently be the same as that obtained by multiplying the measure of AB by the measure of AC; that is, the measure of AB being 4 and the measure of AC 3, the measure of $ACDB$ is 4×3 or 12. (Algebra, Art. 38.)

And *generally*, if the measures of two adjacent sides of a rectangle, supposed to be commensurable, be a and b, then the measure of the rectangle will be ab. (Algebra, Art. 39.)

If all lines were commensurable, then, whatever might be the length of two adjacent sides of a rectangle, we might select the unit of length, so that the measures of the two sides should be whole numbers; and then we might apply the processes of Algebra to establish many Propositions in Geometry by simpler methods than those adopted by Euclid.

Take, for example, the theorem in Book II. Prop. IV.

If all lines were commensurable we might proceed thus —

Let the measure of AC be x,
.... of CB ... y,
Then the measure of AB is $x+y$.

Now $$(x+y)^2 = x^2 + y^2 + 2xy,$$
which proves the theorem.

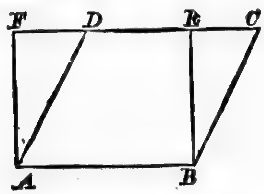
But, inasmuch as all lines are not commensurable, we have in Geometry to treat of *magnitudes* and not of *measures*: that is, when we use the symbol A to represent a line (as in I. 22), A stands for the line itself and not, as in Algebra, for the number of units of length contained by the line.

The method, adopted by Euclid in Book II. to explain the relations between the rectangles contained by certain lines, is more exact than any method founded upon Algebraical principles can be; because his method applies not merely to the case in which the sides of a rectangle are commensurable, but also to the case in which they are incommensurable.

The student is now in a position to understand the practical application of the theory of Equivalence of Areas, of which the foundation is the 35th Proposition of Book I. We shall give a few examples of the use made of this theory in Mensuration.

Area of a Parallelogram.

The area of a parallelogram $ABCD$ is equal to the area of the rectangle $ABEF$ on the same base AB and between the same parallels AB, FC.



Now BE is the altitude of the parallelogram $ABCD$ if AB be taken as the base.

Hence area of ▱ $ABCD$ = rect. AB, BE.

If then the measure of the base be denoted by b,

and .... altitude .... h,

the measure of the area of the ▱ will be denoted by bh

That is, when the base and altitude are commensurable,

measure of area = measure of base into measure of altitude.

Area of a Triangle.

If from one of the angular points A of a triangle ABC, a perpendicular AD be drawn to BC, Fig. 1, or to BC produced, Fig. 2,

Fig. 1.

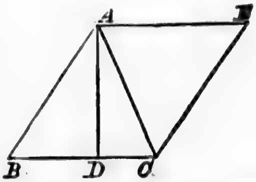
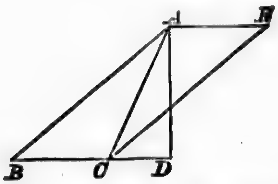


Fig. 2.



and if, in both cases, a parallelogram $ABCE$ be completed of which AB, BC are adjacent sides,

$$\text{area of } \triangle ABC = \text{half of area of } \square ABCE.$$

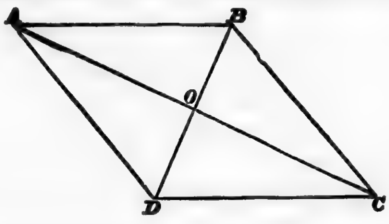
Now if the measure of BC be b,
and ....AD... h,

$$\text{measure of area of } \square ABCE \text{ is } bh;$$

$$\therefore \text{measure of area of } \triangle ABC \text{ is } \frac{bh}{2}.$$

Area of a Rhombus.

Let $ABCD$ be the given rhombus.
Draw the diagonals AC and BD, cutting one another in O.



It is easy to prove that AC and BD bisect each other at right angles.

Then if the measure of AC be x,
and .... BD ... y,

measure of area of rhombus = twice measure of $\triangle ACD$.

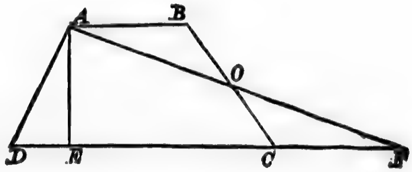
$$= \text{twice } \frac{xy}{4}$$

$$= \frac{xy}{2}.$$

Area of a Trapezium.

Let $ABCD$ be the given trapezium, having the sides AB, CD parallel.

Draw AE at right angles to CD.



Produce DC to F, making $CF=AB$.

Join AF, cutting BC in O.

Then in $\triangle$s AOB, COF,

$\because \angle BAO = \angle CFO$, and $\angle AOB = \angle FOC$, and $AB=CF$;

$$\therefore \triangle COF = \triangle AOB. \qquad \text{I. 26.}$$

Hence trapezium $ABCD = \triangle ADF$.

Now suppose the measures of AB, CD, AE to be m, n, p respectively;

$$\therefore \text{measure of } DF = m+n, \because CF=AB.$$

Then measure of area of trapezium

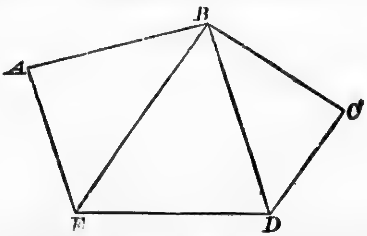
$$= \tfrac{1}{2} (\text{measure of } DF \times \text{measure of } AE)$$
$$= \tfrac{1}{2} (m+n) \times p.$$

That is, the measure of the area of a trapezium is found by multiplying half the measure of the sum of the parallel sides by the measure of the perpendicular distance between the parallel sides.

Area of an Irregular Polygon.

There are three methods of finding the area of an irregular polygon, which we shall here briefly notice.

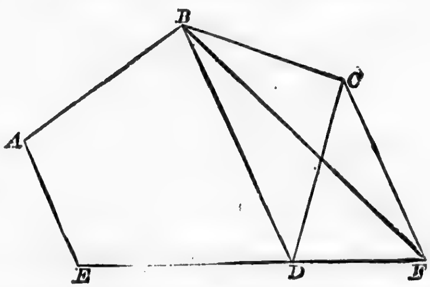
I. *The polygon may be divided into triangles,* and the area of each of these triangles be found separately.



Thus the area of the irregular polygon *ABCDE* is equal to the sum of the areas of the triangles *ABE, EBD, DBC.*

II. *The polygon may be converted into a single triangle of equal area.*

If *ABCDE* be a pentagon, we can convert it into an equivalent quadrilateral by the following process :



Join *BD* and draw *CF* parallel to *BD*, meeting *ED* produced in *F*, and join *BF*.

Then will quadrilateral *ABFE*=pentagon *ABCDE.*

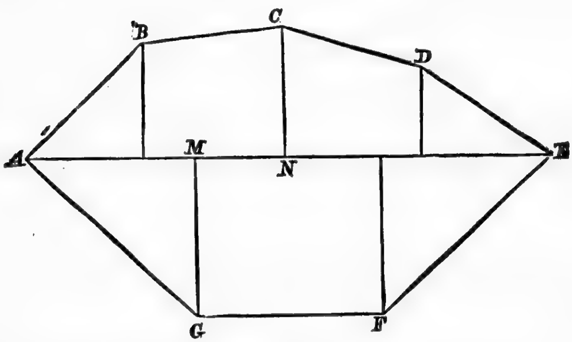
For △ *BDF*= △ *BCD*, on same base *BD* and between same parallels.

If, then, from the pentagon we remove △ *BCD*, and add △ *BDF* to the remainder, we obtain a quadrilateral *ABFE* equivalent to the pentagon *ABCDE*.

The quadrilateral may then, by a similar process, be converted into an equivalent triangle, and thus a polygon of any number of sides may be gradually converted into an equivalent triangle.

The area of this triangle may then be found.

III. The third method is chiefly employed in practice by Surveyors



Let *ABCDEFG* be an irregular polygon.

Draw *AE*, the longest diagonal, and drop perpendiculars on *AE* from the other angular points of the polygon.

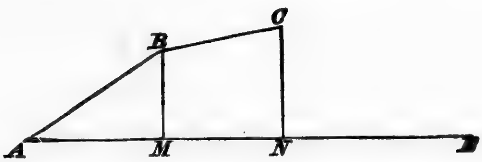
The polygon is thus divided into figures which are either right-angled triangles, rectangles, or trapeziums ; and the areas of each of these figures may be readily calculated.

NOTE 7. *On Projections.*

The projection of a *point* B, on a straight line of unlimited length AE, is the point M at the foot of the perpendicular dropped from B on AE.

The projection of a *straight line* BC, on a straight line of unlimited length AE, is MN,—the part of AE intercepted between perpendiculars drawn from B and C.

When two lines, as AB and AE, form an angle, the projection of AB on AE is AM.



We might employ the term projection with advantage to shorten and make clearer the enunciations of Props. XII. and XIII. of Book II.

Thus the enunciation of Prop. XII. might be :—

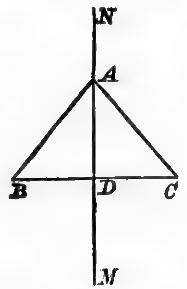
"In oblique-angled triangles, the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the projection of the other on it."

The enunciation of Prop. XIII. might be altered in a similar manner.

NOTE 8. *On Loci.*

Suppose we have to determine the position of a point, which is equidistant from the extremities of a given straight line BC.

There is an infinite number of points satisfying this condition, for the vertex of any isosceles triangle, described on BC as its base, is equidistant from B and C.



Let ABC be *one* of the isosceles triangles described on BC.

If BC be bisected in D, MN, a perpendicular to BC drawn through D, will pass through A.

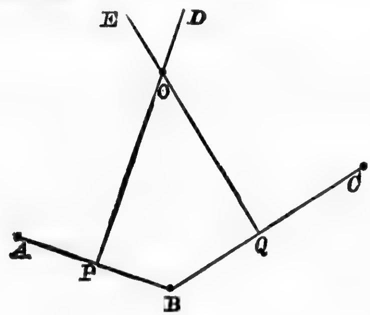
It is easy to shew that any point in MN, or MN produced in either direction, is equidistant from B and C.

It may also be proved that no point out of MN is equidistant from B and C.

The line MN is called the Locus of all the points, infinite in number, which are equidistant from B and C.

DEF. In plane Geometry *Locus* is the name given to a line, straight or curved, all of whose points satisfy a certain geometrical condition (or have a common property), to the exclusion of all other points.

Next, suppose we have to determine the position of a point, which is equidistant from three given points A, B, C, not in the same straight line.



If we join A and B, we know that all points equidistant from A and B lie in the line PD, which bisects AB at right angles.

If we join B and C, we know that all points equidistant from B and C lie in the line QE, which bisects BC at right angles.

Hence O, the point of intersection of PD and QE, is the only point equidistant from A, B and C.

PD is the Locus of points equidistant from A and B,
QE.. B and C,
and the Intersection of these Loci determines the point, which is equidistant from A, B and C.

Examples of Loci.

Find the loci of

(1) Points at a given distance from a given point.

(2) Points at a given distance from a given straight line.

(3) The middle points of straight lines drawn from a given point to a given straight line.

(4) Points equidistant from the arms of an angle.

(5) Points equidistant from a given circle.

(6) Points equally distant from two straight lines which intersect.

NOTE 9. *On the Methods employed in the solution of Problems.*

In the solution of Geometrical Exercises, certain methods may be applied with success to particular classes of questions.

We propose to make a few remarks on these methods, so far as they are applicable to the first two books of Euclid's Elements.

The Method of Synthesis.

In the Exercises, attached to the Propositions in the preceding pages, the construction of the diagram, necessary for the solution of each question, has usually been fully described, or sufficiently suggested.

The student has in most cases been required simply to apply the geometrical fact, proved in the Proposition preceding the exercise, in order to arrive at the conclusion demanded in the question.

This way of proceeding is called Synthesis (*σύνθεσις*=composition), because in it we proceed by a regular chain of reasoning from what is *given* to what is *sought.* This being the method employed by Euclid throughout the Elements, we have no need to exemplify it here.

The Method of Analysis.

The solution of many Problems is rendered more easy *by supposing the problem solved and the diagram constructed.* It is then often possible to observe relations between lines, angles and figures in the diagram, which are suggestive of the steps by which the necessary construction might have been effected.

This is called the Method of Analysis (*ἀνάλυσις*=resolution). It is a method of discovering truth by reasoning concerning things unknown or propositions merely supposed, as if the one were given or the other were really true. The process can best be explained by the following examples.

Our first example of the Analytical process shall be the 31st Proposition of Euclid's First Book.

Ex. 1. *To draw a straight line through a given point parallel to a given straight line.*

Let A be the given point, and BC be the given straight line.

Suppose the problem to be effected, and EF to be the straight line required.

Now we know that any straight line AD drawn from A to meet BC makes equal angles with EF and BC. (I. 29.)

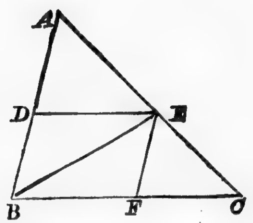
This is a fact from which we can work backward, and arrive at the steps necessary for the solution of the problem; thus:

Take any point D in BC, join AD, make $\angle EAD = \angle ADC$, and produce EA to F: then EF must be parallel to BC.

Ex. 2. *To inscribe in a triangle a rhombus, having one of its angles coincident with an angle of the triangle.*

Let ABC be the given triangle.

Suppose the problem to be effected, and $DBFE$ to be the rhombus.



Then if EB be joined, $\angle DBE = \angle FBE$.

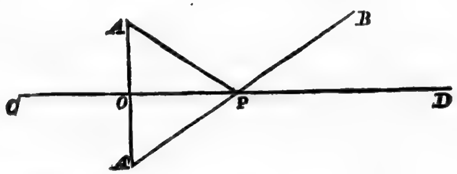
This is a fact from which we can work backward, and deduce the necessary construction; thus:

Bisect $\angle ABC$ by the straight line BE, meeting AC in E.
Draw ED and EF parallel to BC and AB respectively.
Then $DBFE$ is the rhombus required. (See Ex. 4, p. 59.)

Ex. 3. *To determine the point in a given straight line, at which straight lines, drawn from two given points, on the same side of the given line, make equal angles with it.*

Let CD be the given line, and A and B the given points.

Suppose the problem to be effected, and P to be the point required.



We then reason thus :

If BP were produced to some point A',

$\angle CPA'$, being $= \angle BPD$, will be $= \angle APC$.

Again, if PA' be made equal to PA,

AA' will be bisected by CP at right angles.

This is a fact from which we can work backward, and find the steps necessary for the solution of the problem ; thus :

From A draw $AO \perp$ to CD.

Produce AO to A', making $OA' = OA$.

Join BA', cutting CD in P.

Then P is the point required.

Note 10. *On Symmetry.*

The problem, which we have just been considering, suggests the following remarks :

If two points, A and A', be so situated with respect to a straight line CD, that CD bisects at right angles the straight line joining A and A', then A and A' are said to be *symmetrical* with regard to CD.

The importance of symmetrical relations, as suggestive of methods for the solution of problems, cannot be fully shewn

to a learner, who is unacquainted with the properties of the circle. The following example, however, will illustrate this part of the subject sufficiently for our purpose at present.

Find a point in a given straight line, such that the sum of its distances from two fixed points on the same side of the line is a minimum, that is, less than the sum of the distances of any other point in the line from the fixed points.

Taking the diagram of the last example, suppose CD to be the given line, and A, B the given points.

Now if A and A' be symmetrical with respect to CD, we know that *every* point in CD is equally distant from A and A'. (See Note 8, p. 103.)

Hence the sum of the distances of any point in CD from A and B is equal to the sum of the distances of that point from A' and B.

But the sum of the distances of a point in CD from A' and B is the least possible when it lies in the straight line joining A' and B.

Hence the point P, *determined as in the last example*, is the point required.

NOTE. Propositions IX., X., XI., XII. of Book I. give good examples of symmetrical constructions.

NOTE 11. *Euclid's Proof of I. 5.*

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles $\triangle$, having $AB = AC$

Produce AB, AC to D and E.

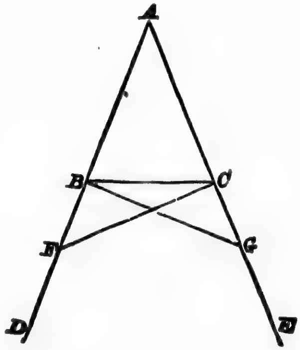
Then must $\angle\, ABC = \angle\, ACB$,

and $\angle\, DBC = \angle\, ECB$.

In BD take any pt. F.

From AE cut off $AG=AF$.

Join FC and GB.



Then in $\triangle$s AFC, AGB,

$\because FA=GA$, and $AC=AB$, and $\angle FAC=\angle GAB$,

$\therefore FC=GB$, and $\angle AFC=\angle AGB$, and $\angle ACF=\angle ABG$. I. 4.

Again, $\because AF=AG$,

of which the parts AB, AC are equal,

$\therefore$ remainder BF=remainder CG. Ax. 3.

Then in $\triangle$s BFC, CGB,

$\because BF=CG$, and $FC=GB$, and $\angle BFC=\angle CGB$,

$\therefore \angle FBC=\angle GCB$, and $\angle BCF=\angle CBG$, I. 4.

Now it has been proved that $\angle ACF=\angle ABG$,

of which the parts $\angle BCF$ and $\angle CBG$ are equal;

$\therefore$ remaining $\angle ACB$=remaining $\angle ABC$. Ax. 3.

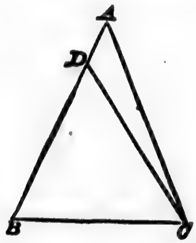
Also it has been proved that $\angle FBC=\angle GCB$,

that is, $\angle DBC=\angle ECB$.

Q. E. D.

Note 12. *Euclid's Proof of I. 6.*

If two angles of a triangle be equal to one another, the sides also, which subtend the equal angles, shall be equal to one another.



In $\triangle ABC$ let $\angle ACB = \angle ABC$.

Then must $AB = AC$.

For if not, AB is either greater or less than AC.

Suppose AB to be greater than AC.

From AB cut off $BD = AC$, and join DC.

Then in $\triangle$s DBC, ACB,

$\because$ $DB = AC$, and BC is common, and $\angle DBC = \angle ACB$,

$\therefore$ $\triangle DBC = \triangle ACB$; I. 4.

that is, the less = the greater; which is absurd.

$\therefore$ AB is not greater than AC.

Similarly it may be shewn that AB is not less than AC;

$\therefore$ $AB = AC$.

Q. E. D.

Note 13. *Euclid's Proof of I. 7.*

Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and their sides which are terminated in the other extremity of the base equal also.

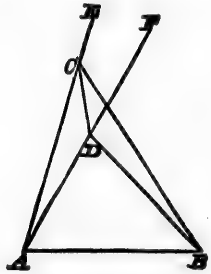
If it be possible, on the same base AB, and on the same side of it, let there be two $\triangle$s ACB, ADB, such that $AC = AD$, and also $BC = BD$.

Join CD.

First, when the vertex of each of the △s is *outside* the other △ (Fig. 1.);

Fig 1.

Fig. 2.



$\because AD = AC,$

$\therefore \angle ACD = \angle ADC.$ I. 5.

But $\angle ACD$ is greater than $\angle BCD$;

$\therefore \angle ADC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again, $\because BC = BD,$

$\therefore \angle BDC = \angle BCD,$

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Secondly, when the vertex D of one of the △s falls *within* the other △ (Fig. 2);

Produce AC and AD to E and F

Then $\because AC = AD.$

$\therefore \angle ECD = \angle FDC.$ I. 5.

But $\angle ECD$ is greater than $\angle BCD$;

$\therefore \angle FDC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again, $\because BC = BD,$

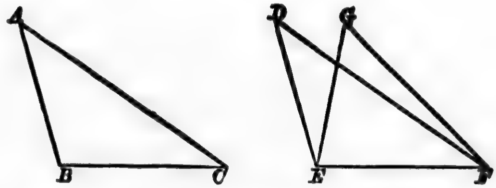
$\therefore \angle BDC = \angle BCD$;

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Lastly, when the vertex D of one of the △s falls on a side BC of the other, it is plain that BC and BD cannot be equal. Q. E. D.

NOTE 14. *Euclid's Proof of I. 8.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one must be equal to the angle contained by the two sides of the other.



Let the sides of the $\triangle$s ABC, DEF be equal, each to each,
that is, $AB = DE$, $AC = DF$ and $BC = EF$.

Then must $\angle BAC = \angle EDF$.

Apply the $\triangle ABC$ to the $\triangle DEF$.
so that pt. B is on pt. E, and BC on EF.

Then $\because BC = EF$,

$\therefore C$ will coincide with F,

and BC will coincide with EF.

Then AB and AC must coincide with DE and DF.

For if AB and AC have a different position, as GE, GF, then upon the same base and upon the same side of it there can be two $\triangle$s, which have their sides which are terminated in one extremity of the base equal, and their sides which are terminated in the other extremity of the base also equal: which is impossible. I. 7.

$\therefore$ since base BC coincides with base EF,

AB must coincide with DE, and AC with DF;

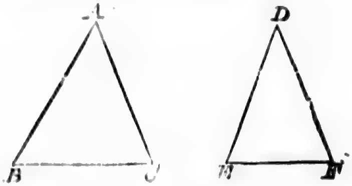
$\therefore \angle BAC$ coincides with and is equal to $\angle EDF$.

Q. E. D.

NOTE 15. *Another Proof of I. 24.*

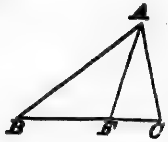
In the $\triangle$s ABC, DEF, let $AB = DE$ and $AC = DF$, and let $\angle BAC$ be greater than $\angle EDF$.

Then must BC be greater than EF.

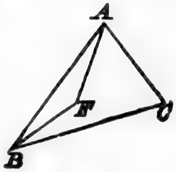


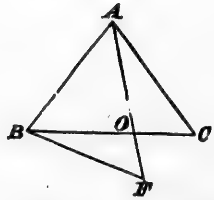
Apply the $\triangle DEF$ to the $\triangle ABC$
so that DE coincides with AB.
Then $\because \angle EDF$ is less than $\angle BAC$,
DF will fall between BA and AC,
and F will fall *on*, or *above*, or *below*, BC.

I. If F fall on BC,
BF is less than BC;
$\therefore$ EF is less than BC.



II. If F fall *above* BC,
BF, FA together are less than
BC, CA,
and $FA = CA$;
$\therefore$ BF is less than BC;
$\therefore$ EF is less than BC.



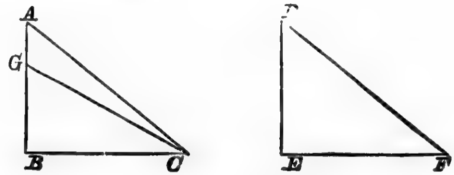


III. If F fall *below* BC,
let AF cut BC in O.

Then BO, OF together are greater than BF, I. 20.
and OC, AO .... AC; I. 20.
$\therefore$ BC, AF .... BF, AC together,
and $AF = AC$,
$\therefore$ BC is greater than BF.
and $\therefore$ EF is less than BC. Q. E. D.

NOTE 16. *Euclid's Proof of I. 26.*

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.



In $\triangle$s ABC, DEF,

Let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$;
and *first*,

Let the sides adjacent to the equal $\angle$s in each be equal, that is, let $BC = EF$.

Then must $AB = DE$, and $AC = DF$, and $\angle BAC = \angle EDF$.

For if AB be not $= DE$, one of them must be the greater.

Let AB be the greater, and make $GB = DE$, and join GC

Then in $\triangle$s GBC, DEF,

$\because GB = DE$, and $BC = EF$, and $\angle GBC = \angle DEF$,

$\therefore \angle GCB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$ by hypothesis;

$\therefore \angle GCB = \angle ACB$;

that is, the less = the greater, which is impossible.

$\therefore AB$ is not greater than DE.

In the same way it may be shewn that AB is not less than DE;

$\therefore AB = DE$.

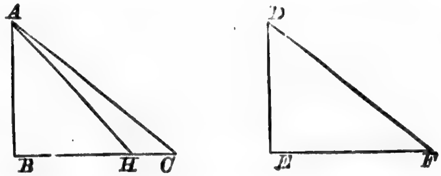
Then in $\triangle$s ABC, DEF,

$\because AB = DE$, and $BC = EF$, and $\angle ABC = \angle DEF$,

$\therefore AC = DF$, and $\angle BAC = \angle EDF$. I. 4.

Next, let the sides which are opposite to equal angles in each triangle be equal, viz., $AB=DE$.

Then must $AC=DF$, *and* $BC=EF$, *and* $\angle BAC = \angle EDF$.



For if BC be not $=EF$, let BC be the greater, and make $BH=EF$, and join AH.

Then in $\triangle$s ABH, DEF,

$\because AB=DE$, and $BH=EF$, and $\angle ABH = \angle DEF$,

$\therefore \angle AHB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$, by hypothesis,

$\therefore \angle AHB = \angle ACB$;

that is, the exterior $\angle$ of $\triangle AHC$ is equal to the interior and opposite $\angle ACB$, which is impossible.

$\therefore BC$ is not greater than EF.

In the same way it may be shewn that BC is not less than EF;

$\therefore BC=EF$.

Then in $\triangle$s ABC, DEF,

$\because AB=DE$, and $BC=EF$, and $\angle ABC = \angle DEF$,

$\therefore AC=DF$, and $\angle BAC = \angle EDF$. I. 4.

Q. E. D.

Miscellaneous Exercises on Books I. and II.

1. *AB* and *CD* are equal straight lines, bisecting one another at right angles. Shew that *ACBD* is a square.

2. From a point in the side of a parallelogram draw a line dividing the parallelogram into two equal parts.

3. In the triangle *FDC*, if *FCD* be a right angle, and angle *FDC* be double of angle *CFD*, shew that *FD* is double of *DC*.

4. If *ABC* be an equilateral triangle, and *AD*, *BE* be perpendiculars to the opposite sides intersecting in *F*; shew that the square on *AB* is equal to three times the square on *AF*.

5. Describe a rhombus, which shall be equal to a given triangle, and have each of its sides equal to one side of the triangle.

6. From a given point, outside a given straight line, draw a line making with the given line an angle equal to a given rectilineal angle.

7. If two straight lines be drawn from two given points to meet in a given straight line, shew that the sum of these lines is the least possible, when they make equal angles with the given line.

8. *ABCD* is a parallelogram, whose diagonals *AC*, *BD* intersect in *O*; shew that if the parallelograms *AOBP*, *DOCQ* be completed, the straight line joining *P* and *Q* passes through *O*.

9. *ABCD*, *EBCF* are two parallelograms on the same base *BC*, and so situated that *CF* passes through *A*. Join *DF*, and produce it to meet *BE* produced in *K*; join *FB*, and prove that the triangle *FAB* equals the triangle *FEK*.

10. The alternate sides of a polygon are produced to meet; shew that all the angles at their points of intersection together with four right angles are equal to all the interior angles of the polygon.

11. Shew that tne perimeter of a rectangle is always greater than that of the square equal to the rectangle.

12. Shew that the opposite sides of an equiangular hexagon are parallel, though they be not equal.

13. If two equal straight lines intersect each other anywhere at right angles, shew that the area of the quadrilateral formed by joining their extremities is invariable, and equal to one-half the square on either line.

14. Two triangles ACB, ADB are constructed on the same side of the same base AB. Shew that if $AC=BD$ and $AD=BC$, then CD is parallel to AB; but if $AC=BC$ and $AD=BD$, then CD is perpendicular to AB.

15. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB, such that DB may be equal to the perpendicular from D on AC.

16. Find the locus of the vertices of triangles of equal area on the same base, and on the same side of it.

17. Shew that the perimeter of an isosceles triangle is less than that of any triangle of equal area on the same base.

18. If each of the equal angles of an isosceles triangle be equal to one-fourth the vertical angle, and from one of them a perpendicular be drawn to the base, meeting the opposite side produced, then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

19. If a straight line terminated by the sides of a triangle be bisected, shew that no other line terminated by the same two sides can be bisected in the same point.

20. Shew how to bisect a given quadrilateral by a straight line drawn from one of its angles.

21. Given the lengths of the two diagonals of a rhombus, construct it.

22. $ABCD$ is a quadrilateral figure: construct a triangle whose base shall be in the line AB, such that its altitude shall be equal to a given line, and its area equal to that of the quadrilateral.

23. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

24. If ABC be a triangle, in which C is a right angle, and DE be drawn from a point D in AC at right angles to AB, prove that the rectangles AB, AE and AC, AD are equal.

25. A line is drawn bisecting parallelogram $ABCD$, and meeting AD, BC in E and F: shew that the triangles EBF, CED are equal.

26. Upon the hypotenuse BC and the sides CA, AB of a right-angled triangle ABC, squares $BDEC$, AF and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

27. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, the second bisecting the base, and the third perpendicular to the base, shew that the first lies, both in position and magnitude, between the other two.

28. If ABC be a triangle, whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

29. Let ACB, ADB be two right-angled triangles having a common hypotenuse AB. Join CD and on CD produced both ways draw perpendiculars AE, BF. Shew that the sum of the squares on CE and CF is equal to the sum of the squares on DE and DF.

30. In the base AC of a triangle take any point D: bisect AD, DC, AB, BC at the points E, F, G, H respectively. Shew that EG is equal and parallel to FH.

31. If AD be drawn from the vertex of an isosceles triangle ABC to a point D in the base, shew that the rectangle BD, DC is equal to the difference between the squares on AB and AD.

32. If in the sides of a square four points be taken at equal distances from the four angular points taken in order, the figure contained by the straight lines, which join them, shall also be a square.

33. If the sides of an equilateral and equiangular pentagon be produced to meet, shew that the sum of the angles at the points of meeting is equal to two right angles.

34. Describe a square that shall be equal to the difference between two given and unequal squares.

35. *ABCD*, *AECF* are two parallelograms, *EA*, *AD* being in a straight line. Let *FG*, drawn parallel to *AC*, meet *BA* produced in *G*. Then the triangle *ABE* equals the triangle *ADG*.

36. From *AC*, the diagonal of a square *ABCD*, cut off *AE* equal to one-fourth of *AC*, and join *BE*, *DE*. Shew that the figure *BADE* is equal to twice the square on *AE*.

37. If *ABC* be a triangle, with the angles at *B* and *C* each double of the angle at *A*, prove that the square on *AB* is equal to the square on *BC* together with the rectangle *AB*, *BC*.

38. If two sides of a quadrilateral be parallel, the triangle contained by either of the other sides and the two straight lines drawn from its extremities to the middle point of the opposite side is half the quadrilateral.

39. Describe a parallelogram equal to and equiangular with a given parallelogram, and having a given altitude.

40. If the sides of a triangle taken in order be produced to twice their original lengths, and the outer extremities be joined, the triangle so formed will be seven times the original triangle.

41. If one of the acute angles of a right-angled isosceles triangle be bisected, the opposite side will be divided by the bisecting line into two parts, such that the square on one will be double of the square on the other.

42. *ABC* is a triangle, right-angled at *B*, and *BD* is drawn perpendicular to the base, and is produced to *E* until *ECB* is a right angle; prove that the square on *BC* is equal to the sum of the rectangles *AD*, *DC* and *BD*, *DE*.

43. Shew that the sum of the squares on two unequal lines is greater than twice the rectangle contained by the lines.

44. From a given isosceles triangle cut off a trapezium, having the base of the triangle for one of its parallel sides, and having the other three sides equal.

45. If any number of parallelograms be constructed having their sides of given length, shew that the sum of the squares on the diagonals of each will be the same.

46. *ABCD* is a right-angled parallelogram, and *AB* is double of *BC*; on *AB* an equilateral triangle is constructed: shew that its area will be less than that of the parallelogram.

47. A point *O* is taken within a triangle *ABC*, such that the angles *BOC*, *COA*, *AOB* are equal; prove that the squares on *BC*, *CA*, *AB* are together equal to the rectangles contained by *OB*, *OC*; *OC*, *OA*; *OA*, *OB*; and twice the sum of the squares on *OA*, *OB*, *OC*.

48. If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

49. *ABC* is a triangle right-angled at *A*; in the hypotenuse two points *D*, *E* are taken such that $BD=BA$ and $CE=CA$; shew that the square on *DE* is equal to twice the rectangle contained by *BE*, *CD*.

50. Given one side of a rectangle which is equal in area to a given square, find the other side.

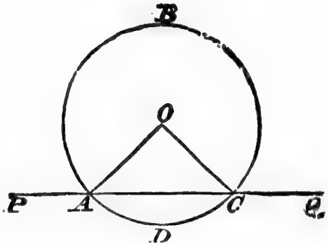
51. *AB*, *AC* are the two equal sides of an isosceles triangle; from *B*, *BD* is drawn perpendicular to *AC*, meeting it in *D*; shew that the square on *BD* is greater than the square on *CD* by twice the rectangle *AD*, *CD*.

BOOK III.

Postulate.

A point is within, or without, a circle, according as its distance from the centre is less, or greater than, the radius of the circle.

Def. I. A straight line, as PQ, drawn so as to cut a circle $ABCD$, is called a Secant.

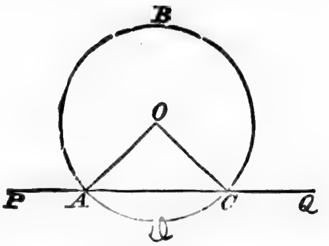


That such a line can only meet the circumference in *two* points may be shewn thus:

Some point within the circle is the centre; let this be O. Join OA. Then (Ex. 1, I. 16) we can draw one, and only one, straight line from O, to meet the straight line PQ, such that it shall be equal to OA. Let this line be OC. Then A and C are the only points in PQ, which are *on* the circumference of the circle.

Def. II. The portion AC of the secant PQ, intercepted by the circle, is called a Chord.

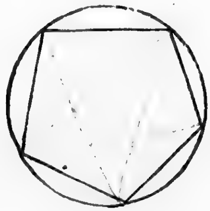
Def. III. The two portions, into which a chord divides the circumference, as ABC and ADC, are called Arcs.



Def. IV. The two figures into which a chord divides the circle, as ABC and ADC, that is, the figures, of which the boundaries are respectively the arc ABC and the chord AC, and the arc ADC and the chord AC, are called Segments of the circle.

Def. V. The figure $AOCD$, whose boundaries are two radii and the arc intercepted by them, is called a Sector.

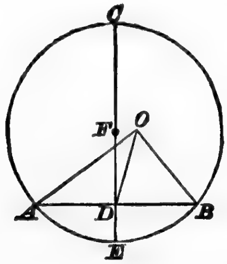
Def. VI. A circle is said to be *described about* a rectilinear figure, when the circumference passes through each of the angular points of the figure.



And the figure is said to be *inscribed* in the circle.

Proposition I. Theorem.

The line, which bisects a chord of a circle at right angles, must contain the centre.



Let ABC be the given ⊙.

Let the st. line CE bisect the chord AB at rt. angles in D.

Then the centre of the ⊙ must lie in CE.

For if not, let O, a pt. out of CE, be the centre; and join OA, OD, OB.

Then, in △s ODA, ODB,

∵ $AD = BD$, and DO is common, and $OA = OB$;

∴ ∠ ODA = ∠ ODB; I. c.

and ∴ ∠ ODB is a right ∠. I. Def. 9

But ∠ CDB is a right ∠, by construction;

∴ ∠ ODB = ∠ CDB, which is impossible;

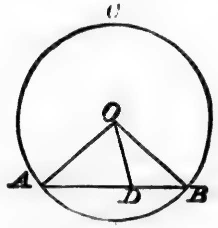
∴ O is not the centre.

Thus it may be shewn that no point, out of CE, can be the centre, and ∴ the centre must lie in CE.

Cor. *If the chord CE be bisected in F, then F is the centre of the circle.*

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line, which joins them, must fall within the circle.



Let A and B be any two pts. in the ○ce of the ⊙ ABC.

Then must the st. line AB fall within the ⊙.

Take any pt. D in the line AB.

Find O the centre of the ⊙. III. 1, Cor.

Join OA, OD, OB.

Then $\because \angle OAB = \angle OBA$, I. A.

and $\angle ODB$ is greater than $\angle OAB$, I. 16.

$\therefore \angle ODB$ is greater than $\angle OBA$;

and $\therefore$ OB is greater than OD. I. 19.

$\therefore$ the distance of D from O is less than the radius of the ⊙,

and $\therefore$ D lies within the ⊙. Post.

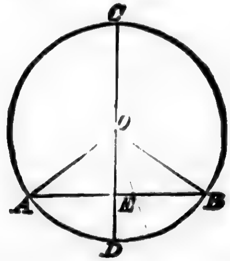
And the same may be shewn of any other pt. in AB.

$\therefore$ AB lies entirely within the ⊙.

Q. E. D.

PROPOSITION III. THEOREM.

If a straight line, drawn through the centre of a circle, bisect a chord of the circle, which does not pass through the centre, it must cut it at right angles: and conversely, if it cut it at right angles, it must bisect it.



In the ⊙ *ABC*, let the chord *AB*, which does not pass through the centre *O*, be bisected in *E* by the diameter *CD*.

Then must CD be ⊥ *to AB.*

Join *OA*, *OB*.

Then in △s *AEO*, *BEO*,

∵ $AE=BE$, and EO is common, and $OA=OB$,

∴ $\angle\ OEA = \angle\ OEB$. I. c.

Hence *OE* is ⊥ to *AB*, I. Def. 9.

that is, *CD* is ⊥ to *AB*.

Next let *CD* be ⊥ to *AB*.

Then must CD bisect AB.

For ∵ $OA=OB$, and OE is common,

in the right-angled △s *AEO*, *BEO*,

∴ $AE=BE$, I. E. Cor. p. 43.

that is, *CD* bisects *AB*. Q. E. D.

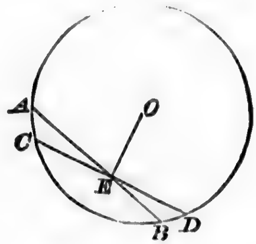
Ex. 1. Shew that, if *CD* does not cut *AB* at right angles, it cannot bisect it.

Ex. 2. A line, which bisects two parallel chords in a circle, is also perpendicular to them.

Ex. 3. Through a given point within a circle, which is not the centre, draw a chord which shall be bisected in that point.

Proposition IV. Theorem.

If in a circle two chords, which do not both pass through the centre, cut one another, they do not bisect each other.



Let the chords AB, CD, which do not both pass through the centre, cut one another, in the pt. E, in the ⊙ $ACBD$.

Then AB, CD do not bisect each other.

If one of them pass through the centre, it is plainly not bisected by the other, which does not pass through the centre.

But if neither pass through the centre, let, if it be possible, $AE = EB$ and $CE = ED$; find the centre O, and join OE.

Then ∵ OE, passing through the centre, bisects AB,

$\therefore \angle OEA$ is a rt. $\angle$. III. 3.

And ∵ OE, passing through the centre, bisects CD,

$\therefore \angle OEC$ is a rt. $\angle$; III. 3

$\therefore \angle OEA = \angle OEC$, which is impossible;

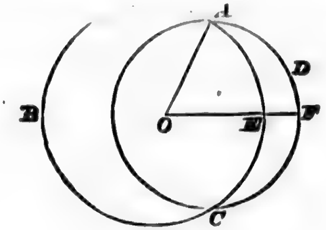
$\therefore AB$, CD do not bisect each other. Q. E. D.

Ex. 1. Shew that the locus of the points of bisection of a[illegible] parallel chords of a circle is a straight line.

Ex. 2. Shew that no parallelogram, except those which are rectangular, can be inscribed in a circle.

Proposition V. Theorem.

If two circles cut one another, they cannot have the same centre.



If it be possible, let O be the common centre of the ⊙s ABC, ADC, which cut one another in the pts. A and C.

Join OA, and draw OEF meeting the ⊙s in E and F.

Then ∵ O is the centre of ⊙ ABC,

$\therefore OE = OA$; I. Def. 13.

and ∵ O is the centre of ⊙ ADC,

$\therefore OF = OA$; I. Def. 13.

$\therefore OE = OF$, which is impossible;

$\therefore O$ is not the common centre.

Q. E. D.

Ex. If two circles cut one another, shew that a line drawn through a point of intersection, terminated by the circumferences and parallel to the line joining the centres, is double of the line joining the centres.

Note. Circles which have the same centre are called *Concentric.*

NOTE 1. *On the Contact of Circles.*

DEF. VII. Circles are said to touch each other, which meet but do not cut each other.

One circle is said to touch another *internally*, when one point of the circumference of the former lies *on*, and no point *without*, the circumference of the other.

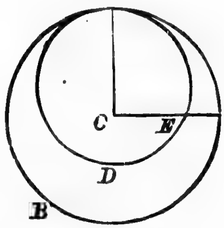
Hence for internal contact one circle must be smaller than the other.

Two circles are said to touch *externally*, when one point of the circumference of the one lies *on*, and no point *within* the circumference of the other.

N.B. No restriction is placed by these definitions on the number of points of contact, and it is not till we reach Prop. XIII. that we prove that there can be *but one point of contact.*

Proposition VI. Theorem.

If one circle touch another internally, they cannot have the same centre.



Let ⊙ ADE touch ⊙ ABC internally,

and let A be a point of contact.

Then *some* point E in the ○ce ADE lies *within* ⊙ ABC.

Def. 7.

If it be possible, let O be the common centre of the two ⊙s.

Join OA, and draw OEC, meeting the ○ces in E and C.

Then ∵ O is the the centre of ⊙ ABC,

$\therefore\ OA = OC$; I. Def. 13.

and ∵ O is the centre of ⊙ ADE,

$\therefore\ OA = OE$. I. Def. 13.

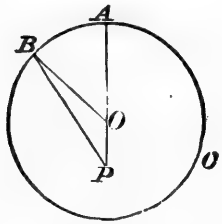
Hence $OE = OC$, which is impossible ,

$\therefore$ O is not the common centre of the two ⊙s

Q. E. D.

Proposition VII. Theorem.

If from any point within a circle, which is not the centre, straight lines be drawn to the circumference, the greatest of these lines is that which passes through the centre.



Let ABC be a ⊙, of which O is the centre.

From P, any pt. within the ⊙, draw the st. line PA, passing through O and meeting the ○ce in A.

Then must PA be greater than any other st. line, drawn from P to the ○ce.

For let PB be any other st. line, drawn from P to meet the ○ce in B, and join BO.

Then $\because AO = BO$,

$\therefore AP =$ sum of BO and OP.

But the sum of BO and OP is greater than BP, I. 20.

and $\therefore AP$ is greater than BP. Q. E. D.

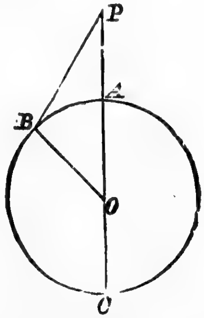
Ex. 1. If AP be produced to meet the circumference in D, shew that PD is less than any other straight line that can be drawn from P to the circumference.

Ex. 2. Shew that PB continually decreases, as B passes from A to D.

Ex. 3. Shew that two straight lines, but not three, that shall be equal, can be drawn from P to the circumference.

Proposition VIII. Theorem.

If from any point without a circle straight lines be drawn to the circumference, the least of these lines is that which, when produced, passes through the centre, and the greatest is that which passes through the centre.



Let ABC be a ⊙, of which O is the centre.

From P any pt. outside the ⊙, draw the st. line $PAOC$, meeting the ○ce in A and C.

Then must PA be less, and PC greater, than any other st. line drawn from P to the ○ce.

For let PB be any other st. line drawn from P to meet the ○ce in B, and join BO.

Then ∵ sum of PB and BO is greater than OP, I. 20.

∴ sum of PB and BO is greater than sum of AP and AO.

But $BO=AO$;

∴ PB is greater than AP.

Again ∵ PB is less than the sum of PO, OB, I. 20.

∴ PB is less than the sum of PO, OC;

∴ PB is less than PC. Q. E. D.

Ex. 1. Shew that PB continually increases as B passes from A to C.

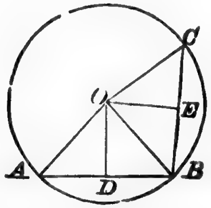
Ex. 2. Shew that from P two straight lines, but not three, that shall be equal, can be drawn to the circumference.

Note. From Props. vii. and viii. we deduce the following Corollary, which we shall use in the proof of Props. xi. and xiii.

Cor. *If a point be taken, within or without a circle, of all straight lines drawn from it to the circumference, the greatest is that which meets the circumference after passing through the centre.*

Proposition IX. Theorem.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.



Let O be a pt. in the ⊙ ABC from which more than two st. lines OA, OB, OC, drawn to the ○ce, are equal.

Then must O be the centre of the ⊙.

Join AB, BC, and draw OD, OE $\perp$ to AB, BC.

Then $\because$ $OA=OB$, and OD is common,

in the right-angled $\triangle$s AOD, BOD,

$\therefore$ $AD=DB$; I. E. Cor. p. 43.

$\therefore$ the centre of the ⊙ is in DO. III. 1.

Similarly it may be shown that

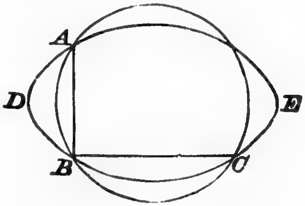
the centre of the ⊙ is EO;

$\therefore$ O is the centre of the ⊙.

Q. E. D.

Proposition X. Theorem.

Two circles cannot have more than two points common to both, without coinciding entirely.



If it be possible, let ABC and ADE be two ⊙s which have more than two pts. in common, as A, B, C.

Join AB, BC.

Then ∵ AB is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects AB at right angles ; III. 1.

and ∵ BC is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects BC at right angles. III. 1.

∴ the centre of each circle is the point, in which the two straight lines, which bisect AB and BC at right angles, meet.

∴ the ⊙s ABC, ADE have a common centre, which is impossible ; III. 5 and 6.

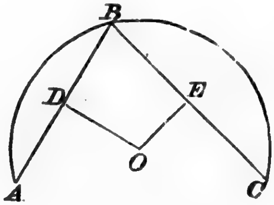
∴ two ⊙s cannot have more than two pts. common to both.

Q. E. D.

Note. We here insert two Propositions, Eucl. III. 25 and IV. 5, which are closely connected with Theorems I. and X. of this book. The learner should compare with this portion of the subject the note on Loci, p. 103.

PROPOSITION A. PROBLEM. (Eucl. III. 25.)

An arc of a circle being given, to complete the circle of which it is a part.



Let ABC be the given arc.

It is required to complete the ⊙ of which ABC is a part.

Take B, any pt. in arc ABC, and join AB, BC.

From D and E, the middle pts. of AB and BC,

draw DO, EO, ⊥s to AB, BC, meeting in O.

Then ∵ AB is to be a chord of the ⊙,

∴ centre of the ⊙ lies in DO; III. 1.

and ∵ BC is to be a chord of the ⊙,

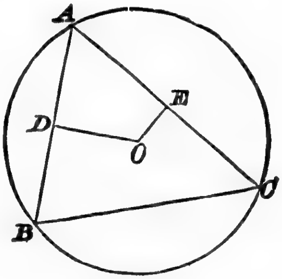
∴ centre of the ⊙ lies in EO. III. 1.

Hence O is the centre of the ⊙ of which ABC is an arc, and if a ⊙ be described, with centre O and radius OA, this will be the ⊙ required.

Q. E. F.

PROPOSITION B. PROBLEM. (Eucl. IV. 5.)

To describe a circle about a given triangle.



Let ABC be the given $\triangle$.

It is required to describe a $\odot$ about the $\triangle$.

From D and E, the middle pts. of AB and AC, draw DO, EO, $\perp$s to AB, AC, and let them meet in O.

Then $\because$ AB is to be a chord of the $\odot$,

$\therefore$ centre of the $\odot$ lies in DO. III. 1.

And $\because$ AC is to be a chord of the $\odot$,

$\therefore$ centre of the $\odot$ lies in EO. III. 1.

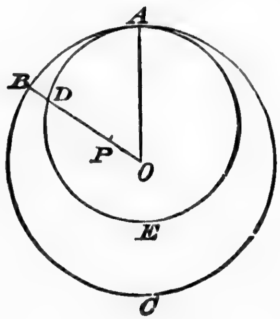
Hence O is the centre of the $\odot$ which can be described about the $\triangle$, and if a $\odot$ be described with centre O and radius OA, this will be the $\odot$ required.

Q. E. F.

Ex. If BAC be a right angle, show that O will coincide with the middle point of BC.

PROPOSITION XI. THEOREM.

If one circle touch another internally at any point, the centre of the interior circle must lie in that radius of the other circle which passes through that point of contact.



Let the ⊙ ADE touch the ⊙ ABC internally, and let A be a pt. of contact.

Find O the centre of ⊙ ABC, and join OA.

Then must the centre of ⊙ ADE lie in the radius OA.

For if not, let P be the centre of ⊙ ADE.

Join OP, and produce it to meet the ○ces in D and B.

Then ∵ P is the centre of ⊙ ADE, and from O are drawn to the ○ce of ADE the st. lines OA, OD, of which OD passes through P,

∴ OD is greater than OA. III. 8, Cor.

But $OA = OB$;

∴ OD is greater than OB,

which is impossible.

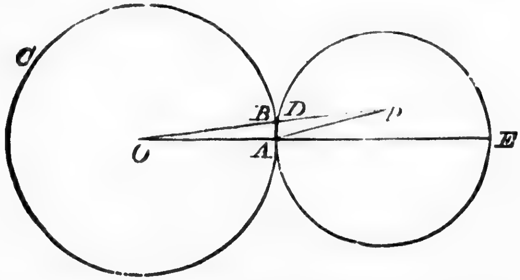
∴ the centre of ⊙ ADE is not out of the radius OA.

∴ it lies in OA.

Q. E. D.

Proposition XII. Theorem.

If two circles touch one another externally at any point, the straight line joining the centre of one with that point of contact must when produced pass through the centre of the other.



Let ⊙ ABC touch ⊙ ADE externally at the pt. A.

Let O be the centre of ⊙ ABC.

Join OA, and produce it to E.

Then must the centre of ⊙ ADE lie in AE.

For if not, let P be the centre of ⊙ ADE.

Join OP meeting the ⊙s in B, D; and join AP.

Then ∵ $OB=OA$,

and $PD=AP$,

∴ OB and PD together $=OA$ and AP together;

∴ OP is not less than OA and AP together.

But OP is less than OA and AP together, I. 20.

which is impossible;

∴ the centre of ⊙ ADE cannot lie out of AE.

Q. E. D.

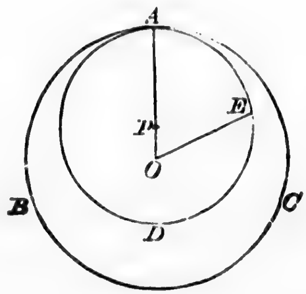
Ex. Three circles touch one another externally, whose centres are A, B, C. Shew that the difference between AB and AC is half as great as the difference between the diameters of the circles, whose centres are B and C.

Proposition XIII. Theorem.

One circle cannot touch another at more points than one, whether it touch it internally or externally.

First let the ⊙ ADE touch the ⊙ ABC *internally* at pt. A.

Then there can be no other point of contact.



Take O the centre of ⊙ ABC

Then P, the centre of ⊙ ADE, lies in OA. III. 11.

Take any pt. E in the ○ce of the ⊙ ADE, and join OE.

Then ∵ from O, a pt. within or without the ⊙ ADE, two lines OA, OE are drawn to the ○ce, of which OA passes through the centre P,

∴ OA is greater than OE, III. 8, Cor.

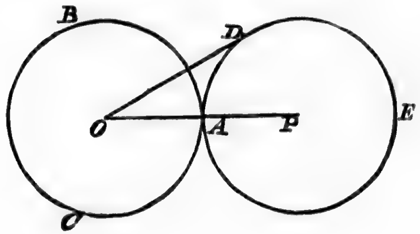
and ∴ E is a point *within* the ⊙ ABC. Post.

Similarly it may be shewn that every pt. of the ○ce of the ⊙ ADE, except A, lies *within* the ⊙ ABC;

∴ A is the only point at which the ⊙s meet.

Next, let the ⊙s *ABC*, *ADE* touch *externally* at the pt. *A*.

Then there can be no other point of contact.



Take *O* the centre of the ⊙ *ABC*.

Then *P*, the centre of the ⊙ *ADE*, lies in *OA* produced.
III. 12.

Take any pt. *D* in the ○ce of the ⊙ *ADE*, and join *OD*.

Then ∵ from *O*, a pt. without the ⊙ *ADE*, two lines *OA*, *OD* are drawn to the ○ce, of which *OA* when produced passes through the centre *P*,

∴ *OD* is greater than *OA*; III. 8.

∴ *D* is a point *without* the ⊙ *ABC*. Post.

Similarly, it may be shewn that every pt. of the ○ce of *ADE*, except *A*, lies *without* the ⊙ *ABC*;

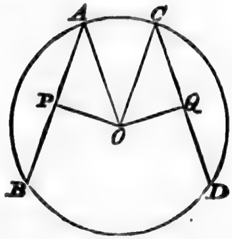
∴ *A* is the only point at which the ⊙s meet.

Q. E. D.

Def. VIII. The distance of a chord from the centre is measured by the length of the perpendicular drawn from the centre to the chord.

Proposition XIV. Theorem.

Equal chords in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.



Let the chords AB, CD in the $\odot$ $ABDC$ be equal.

Then must AB and CD be equally distant from the centre O.

Draw OP and OQ $\perp$ to AB and CD; and join AO, CO.

Then P and Q are the middle pts. of AB and CD: III. 3.

and $\because AB=CD$, $\therefore AP=CQ$.

Then $\because AP=CQ$, and $AO=CO$,

in the right-angled $\triangle$s AOP, COQ,

$\therefore OP=OQ$; I. E. Cor. p. 43.

and $\therefore$ AB and CD are equally distant from O. Def. 8.

Next, let AB and CD be equally distant from O.

Then must $AB=CD$.

For $\because OP=OQ$, and $AO=CO$,

in the right-angled $\triangle$s AOP, COQ,

$\therefore AP=CQ$, I. E. Cor.

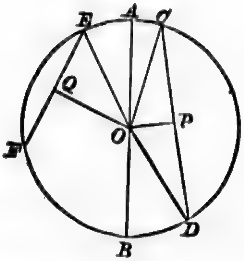
and $\therefore AB=CD$.

Q. E. D.

Ex. In a circle, whose diameter is 10 inches, a chord is drawn, which is 8 inches long. If another chord be drawn, at a distance of 3 inches from the centre, shew whether it is equal or not to the former.

Proposition XV. Theorem.

The diameter is the greatest chord in a circle, and of all others that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.



Let AB be a diameter of the $\odot ABDC$, whose centre is O, and let CD be any other chord, not a diameter, in the $\odot$, nearer to the centre than the chord EF.

Then must AB be greater than CD, and CD greater than EF.

Draw OP, OQ $\perp$ to CD and EF; and join OC, OD, OE.

Then $\because AO=CO$, and $OB=OD$, — I. Def. 13.

$\therefore AB=$sum of CO and OD,

and $\therefore AB$ is greater than CD. — I. 20.

Again, $\because CD$ is nearer to the centre than EF,

$\therefore OP$ is less than OQ. — Def. 8.

Now $\because$ sq. on $OC=$sq. on OE,

$\therefore$ sum of sqq. on OP, $PC=$sum of sqq on OQ, QE. — I. 47.

But sq. on OP is less than sq. on OQ;

$\therefore$ sq. on PC is greater than sq. on QE;

$\therefore PC$ is greater than QE;

and $\therefore CD$ is greater than EF.

Next, let CD be greater than EF.

Then must CD be nearer to the centre than EF.

For $\because CD$ is greater than EF,

$\therefore PC$ is greater than QE.

Now the sum of sqq. on OP, PC=sum of sqq. on OQ, QE.
But sq. on PC is greater than sq. on QE;
$\therefore$ sq. on OP is less than sq. on OQ;
$\therefore$ OP is less than OQ;
and $\therefore$ CD is nearer to the centre than EF.

Q. E. D.

Ex. 1. Draw a chord of given length in a given circle, which shall be bisected by a given chord.

Ex. 2. If two isosceles triangles be of equal altitude, and the sides of one be equal to the sides of the other, shew that their bases must be equal.

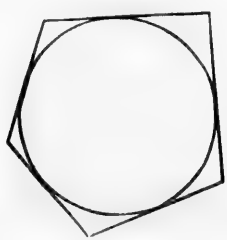
Ex. 3. Any two chords of a circle, which cut a diameter in the same point and at equal angles, are equal to one another.

Def. IX. *A straight line is said to be a* Tangent *to, or to touch, a circle, when it meets and, being produced, does not cut the circle.*

From this definition it follows that the tangent meets the circle in one point only, for if it met the circle in two points it would cut the circle, since the line joining two points in the circumference is, being produced, a secant. (III. 2.)

Def. X. If from any point in a circle a line be drawn at right angles to the tangent at that point, the line is called a Normal to the circle at that point.

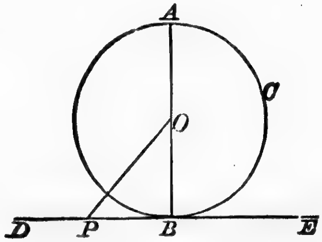
Def. XI. A rectilinear figure is said to be *described about* a circle, when each side of the figure touches the circle.



And the circle is said to be *inscribed* in the figure.

Proposition XVI. Theorem.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, is a tangent to the circle.



Let ABC be a ⊙, of which the centre is O, and the diameter AOB.

Through B draw DE at right angles to AOB. I. 11.

Then must DE be a tangent to the ⊙.

Take any point P in DE, and join OP.

Then, ∵ ∠ OBP is a right angle,

∴ ∠ OPB is less than a right angle, I. 17.

and ∴ OP is greater than OB. I. 19.

Hence P is a point without the ⊙ ABC. Post.

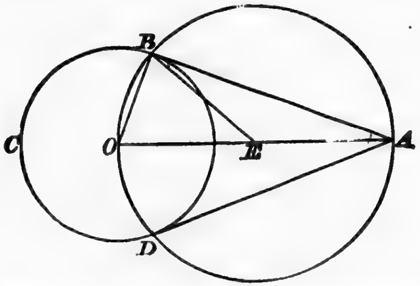
In the same way it may be shewn that every point in DE, or DE produced in either direction, except the point B, lies without the ⊙;

∴ DE is a tangent to the ⊙. Def. 9.

Q. E. D.

Proposition XVII. Problem.

To draw a straight line from a given point, either WITHOUT *or* ON *the circumference, which shall touch a given circle.*



Let A be the given pt., *without* the ⊙ BCD.

Take O the centre of ⊙ BCD, and join OA.

Bisect OA in E, and with centre E and radius EO describe ⊙ $ABOD$, cutting the given ⊙ in B and D.

Join AB, AD. These are tangents to the ⊙ BCD.

Join BO, BE.

Then $\because OE=BE$, $\therefore \angle OBE = \angle BOE$; I. A.

$\therefore \angle AEB =$ twice $\angle OBE$; I. 32.

and $\because AE=BE$, $\therefore \angle ABE = \angle BAE$; I. A.

$\therefore \angle OEB =$ twice $\angle ABE$; I. 32.

$\therefore$ sum of $\angle$s AEB, OEB = twice sum of $\angle$s OBE, ABE, that is, two right angles = twice $\angle OBA$;

$\therefore \angle OBA$ is a right angle,

and $\therefore AB$ is a tangent to the ⊙ BCD. III. 16.

Similarly it may be shewn that AD is a tangent to ⊙ BCD.

Next, let the given pt. be *on* the ○ce of the ⊙, as B.

Then, if BA be drawn ⊥ to the radius OB,

BA is a tangent to the ⊙ at B. III. 16.

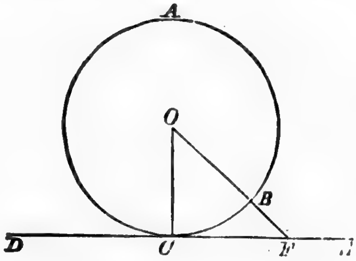
Q. E. D.

Ex. 1. Shew that the two tangents, drawn from a point without the circumference to a circle, are equal.

Ex. 2. If a quadrilateral $ABCD$ be described about a circle, shew that the sum of AB and CD is equal to the sum of AD and BC.

PROPOSITION XVIII. THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact must be perpendicular to the line touching the circle.



Let the st. line DE touch the $\odot$ ABC in the pt. C.

Find O the centre, and join OC.

Then must OC be $\perp$ to DE.

For if it be not, draw $OBF \perp$ to DE, meeting the ○ce in B.

Then $\because \angle OFC$ is a rt. angle,

$\therefore \angle OCF$ is less than a rt. angle, I. 17.

and $\therefore OC$ is greater than OF. I. 19.

But $OC = OB$,

$\therefore OB$ is greater than OF, which is impossible;

$\therefore OF$ is not $\perp$ to DE, and in the same way it may be shewn that no other line drawn from O, but OC, is $\perp$ to DE,

$\therefore OC$ is $\perp$ to DE.

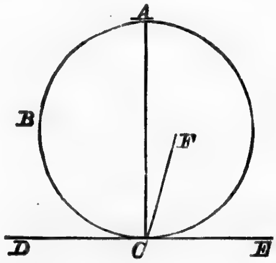
Q. E. D.

Ex. If two straight lines intersect, the centres of all circles touched by both lines lie in two lines at right angles to each other.

NOTE. Prop. XVIII. might be stated thus:—*All radii of a circle are normals to the circle at the points where they meet the circumference.*

PROPOSITION XIX. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle must be in that line.



Let the st. line DE touch the ⊙ ABC at the pt. C, and from C let CA be drawn ⊥ to DE.

Then must the centre of the ⊙ be in CA.

For if not, let F be the centre, and join FC.

Then ∵ DCE touches the ⊙, and FC is drawn from centre to pt. of contact,

∴ ∠ FCE is a rt. angle. III. 18.

But ∠ ACE is a rt. angle.

∴ ∠ FCE = ∠ ACE, which is impossible.

In the same way it may be shewn that no pt. out of CA can be the centre of the ⊙;

∴ the centre of the ⊙ lies in CA.

Q. E. D.

Ex. Two concentric circles being described, if a chord of the greater touch the less, the parts of the chord, intercepted between the two circles, are equal.

NOTE. Prop. XIX. might be stated thus :—*Every normal to a circle passes through the centre.*

Proposition XX. Theorem.

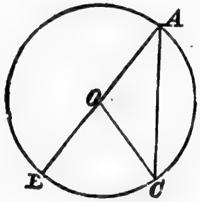
The angle at the centre of a circle is double of the angle at the circumference, subtended by the same arc.

Let ABC be a ⊙, O the centre,

BC any arc, A any pt. in the ○ce.

Then must $\angle BOC =$ *twice* $\angle BAC$.

First, suppose O to be in one of the lines containing the $\angle BAC$.



Then $\because OA = OC$,

$\therefore \angle OCA = \angle OAC$; I. A.

$\therefore$ sum of $\angle$s OCA, $OAC =$ twice $\angle OAC$.

But $\angle BOC =$ sum of $\angle$s OCA, OAC, I. 32.

$\therefore \angle BOC =$ twice $\angle OAC$.

that is, $\angle BOC =$ twice $\angle BAC$

Next, suppose O to be within (fig 1), or without (fig. 2) the $\angle BAC$.

Fig. 1.

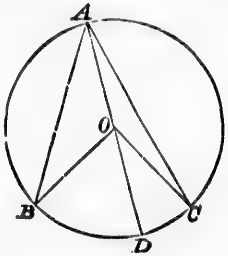
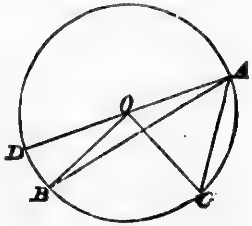


Fig. 2.



Join AO, and produce it to meet the Oce in D.

Then, as in the first case,

$$\angle COD = \text{twice} \angle CAD,$$

$$\text{and} \angle BOD = \text{twice} \angle BAD;$$

$\therefore$, fig. 1, sum of $\angle$s COD, BOD = twice sum of $\angle$s CAD, BAD,

$$\text{that is, } \angle BOC = \text{twice} \angle BAC.$$

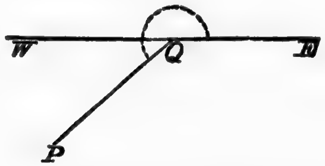
And, fig. 2, difference of $\angle$s COD, BOD = twice difference of $\angle$s CAD, BAD, that is, $\angle BOC$ = twice $\angle BAC$.

Q. E. D.

Ex. From any point in a straight line, touching a circle, a straight line is drawn through the centre, and is terminated by the circumference; the angle between these two straight lines is bisected by a straight line, which intersects the straight line joining their extremities. Shew that the angle between the last two lines is half a right angle.

NOTE 2. *On Flat and Reflex Angles.*

We have already explained (Note 3, Book I., p. 28) how Euclid's definition of an angle may be extended with advantage, so as to include the conception of an angle equal to two right angles: and we now proceed to shew how the Definition given in that Note may be extended, so as to embrace angles greater than two right angles.



Let WQ be a straight line, and QE its continuation.

Then, by the Definition, the angle made by WQ and QE, which we propose to call a FLAT ANGLE, is equal to two right angles.

Now suppose QP to be a straight line, which revolves about the fixed point Q, and which at first coincides with QE.

When QP, revolving from right to left, coincides with QW, it has described an angle equal to two right angles.

When QP has continued its revolution, so as to come into the position indicated in the diagram, it has described an angle EQP, indicated by the dotted line, greater than two right angles, and this we call a REFLEX ANGLE.

To assist the learner, we shall mark these angles with dotted lines in the diagrams.

Admitting the existence of angles, equal to and greater than two right angles, the Proposition last proved may be extended, as we now proceed to shew,

Proposition C. Theorem.

The angle, not less than two right angles, at the centre of a circle is double of the angle at the circumference, subtended by the same arc.

Fig. 1.

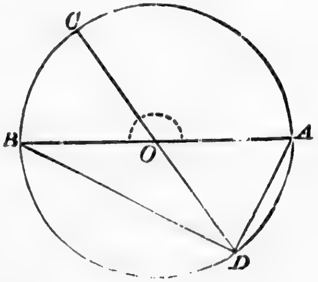
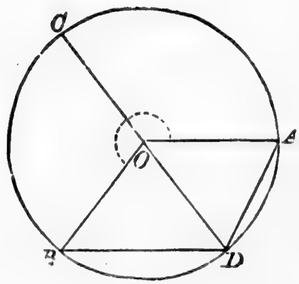


Fig. 2.



In the ⊙ $ACBD$, let the angles AOB (not less than two right angles) at the centre, and ADB at the circumference, be subtended by the same arc ACB.

Then must $\angle\, AOB =$ *twice* $\angle\, ADB$.

Join DO, and produce it to meet the arc ACB in C.

Then $\because \angle\, AOC =$ twice $\angle\, ADO$, III. 20.

and $\angle\, BOC =$ twice $\angle\, BDO$, III. 20.

$\therefore$ sum of $\angle$ s AOC, $BOC =$ twice sum of $\angle$ s ADO, BDO,

that is, $\angle\, AOB =$ twice $\angle\, ADB$.

Q. E. D.

Note. In fig. 1, $\angle\, AOB$ is drawn a flat angle,

and in fig. 2, $\angle\, AOB$ is drawn a reflex angle.

Def. XII. The angle in a segment is the angle contained by two straight lines drawn from any point in the arc to the extremities of the chord.

PROPOSITION XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Fig. 1.

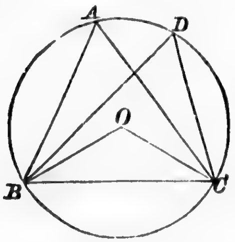
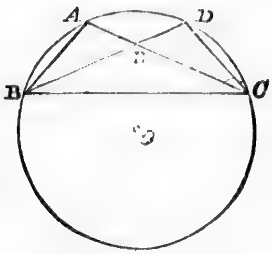


Fig. 2.



Let BAC, BDC be angles in the same segment $BADC$.

Then must $\angle BAC = \angle BDC$.

First, when segment $BADC$ is greater than a semicircle,

From O, the centre, draw OB, OC. (Fig. 1.)

Then, $\because \angle BOC =$ twice $\angle BAC$, III. 20.

and $\angle BOC =$ twice $\angle BDC$, III. 20.

$\therefore \angle BAC = \angle BDC$.

Next, when segment $BADC$ is less than a semicircle,

Let E be the pt. of intersection of AC, DB. (Fig. 2.)

Then $\because \angle ABE = \angle DCE$, by the first case,

and $\angle BEA = \angle CED$, I. 15.

$\therefore \angle EAB = \angle EDC$, I. 32.

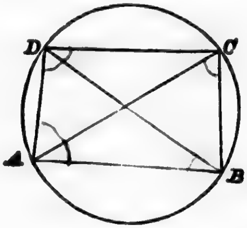
that is, $\angle BAC = \angle BDC$. Q. E. D.

Ex. 1. Shew that, by assuming the possibility of an angle being greater than two right angles, both the cases of this proposition may be included in one.

Ex. 2. If two straight lines, whose extremities are in the circumference of a circle, cut one another, the triangles formed by joining their extremities are equiangular to each other.

Proposition XXII. Theorem.

The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.



Let *ABCD* be a quadrilateral fig. inscribed in a ⊙.

Then must each pair of its opposite $\angle$ *s be together equal to two rt.* $\angle$ *s.*

Draw the diagonals *AC*, *BD*.

Then $\because \angle ADB = \angle ACB$, in the same segment, III. 21.

and $\angle BDC = \angle BAC$, in the same segment; III. 21.

$\therefore$ sum of $\angle$ s *ADB*, *BDC*=sum of $\angle$ s *ACB*, *BAC*;

that is, $\angle ADC$=sum of $\angle$ s *ACB*, *BAC*.

Add to each $\angle ABC$.

Then $\angle$ s *ADC*, *ABC* together=sum of $\angle$ s *ACB*, *BAC*, *ABC*;

and $\therefore \angle$ s *ADC*, *ABC* together=two right $\angle$ s. I. [illegible]

Similarly, it may be shewn,

that $\angle$ s *BAD*, *BCD* together=two right $\angle$ s.

Q. E. D.

Note.—Another method of proving this proposition is given on page 177.

Ex. 1. If one side of a quadrilateral figure inscribed in a circle be produced, the exterior angle is equal to the opposite angle of the quadrilateral.

Ex. 2. If the sides *AB*, *DC* of a quadrilateral inscribed in a circle be produced to meet in *E*, then the triangles *EBC*, *EAD* will be equiangular.

Ex. 3. Shew that a circle cannot be described about a rhombus.

Ex. 4. The lines, bisecting any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.

Ex. 5. *AB*, a chord of a circle, is the base of an isosceles triangle, whose vertex *C* is without the circle, and whose equal sides meet the circle in *D*, *E*: shew that *CD* is equal to *CE*.

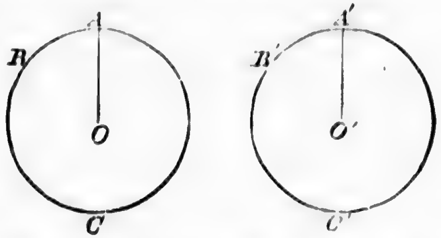
Ex. 6. If in any quadrilateral the opposite angles be together equal to two right angles, a circle may be described about that quadrilateral.

Propositions XXIII. and XXIV., not being required in the method adopted for proving the subsequent Propositions in this book, are removed to the Appendix. Proposition XXV. has been already proved.

NOTE 3. *On the Method of Superposition, as applied to Circles.*

In Props. XXVI. XXVII. XXVIII. XXIX. we prove certain relations existing between chords, arcs, and angles in equal circles. As we shall employ the Method of Superposition, we must state the principles which render this method applicable, as a test of equality, in the case of figures with *circular* boundaries.

DEF. XIII. *Equal circles are those, of which the radii are equal.*



For suppose ABC, $A'B'C'$ to be circles, of which the radii are equal.

Then if $\odot\ A'B'C'$ be applied to $\odot\ ABC$, so that O', the centre of $A'B'C'$, coincides with O, the centre of ABC, it is evident that any *particular* point A' in the $\bigcirc$ce of the former must coincide with *some* point A in $\bigcirc$ce of the latter, because of the equality of the radii $O'A'$ and OA.

Hence $\bigcirc$ce $A'B'C'$ must coincide with $\bigcirc$ce ABC,

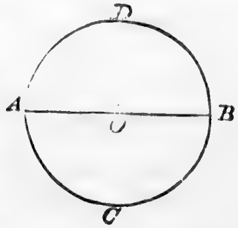
that is, $\odot\ A'B'C' = \odot\ ABC$.

Further, when we have applied the circle $A'B'C'$ to the circle ABC, so that the centres coincide, we may imagine ABC to remain fixed, while $A'B'C'$ revolves round the common centre. Hence we may suppose any particular point B' in the circumference of $A'B'C'$ to be made to coincide with any particular point B in the circumference of ABC.

Again, any radius $O'A'$ of the circle $A'B'C'$ may be made to coincide with any radius OA of the circle ABC.

Also, if $A'B'$ and AB be equal arcs, they may be made to coincide.

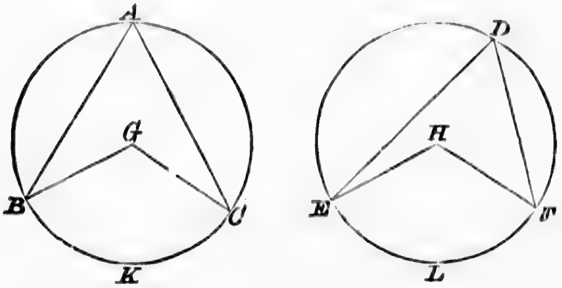
Again, every diameter of a circle divides the circle into equal segments.



For let AOB be a diameter of the circle $ACBD$, of which O is the centre. Suppose the segment ACB to be applied to the segment ADB, so as to keep AB a common boundary: then the arc ACB must coincide with the arc ADB, because every point in each is equally distant from O.

PROPOSITION XXVI. THEOREM.

In equal circles, the arcs, which subtend equal angles, whether they be at the centres or at the circumferences, must be equal.



Let ABC, DEF be equal circles, and let $\angle$s BGC, EHF at their centres, and $\angle$s BAC, EDF at their $\bigcirc$ces, be equal.

Then must arc BKC = *arc* ELF.

For, if $\odot$ ABC be applied to $\odot$ DEF,

so that G coincides with H, and GB falls on HE,

then, $\because GB = HE$, $\therefore B$ will coincide with E.

And $\because \angle BGC = \angle EHF$, $\therefore GC$ will fall on HF;

and $\because GC = HF$, $\therefore C$ will coincide with F.

Then $\because B$ coincides with E and C with F,

$\therefore$ arc BKC will coincide with and be equal to arc ELF.

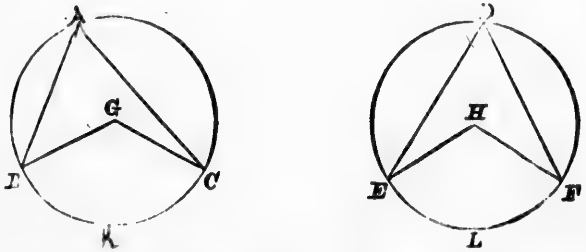
Q. E. D.

COR. Sector $BGCK$ is equal to sector $EHFL$.

NOTE. This and the three following Propositions are, and will hereafter be assumed to be, true for *the same circle* as well as for *equal circles*.

PROPOSITION XXVII. THEOREM.

In equal circles, the angles, which are subtended by equal arcs, whether they are at the centres or at the circumferences, must be equal.



Let *ABC*, *DEF* be equal circles, and let ∠ s *BGC*, *EHF* at their centres, and ∠ s *BAC*, *EDF* at their ○ces, be subtended by equal arcs *BKC*, *ELF*.

Then must ∠ *BGC*= ∠ *EHF, and* ∠ *BAC*= ∠ *EDF*.

For, if ⊙ *ABC* be applied to ⊙ *DEF*,
so that *G* coincides with *H*, and *GB* falls on *HE*,
then ∵ *GB*=*HE*, ∴ *B* will coincide with *E*;
and ∵ arc *BKC*=arc *ELF*, ∴ *C* will coincide with *F*.
Hence, *GC* will coincide with *HF*.
Then ∵ *BG* coincides with *EH*, and *GC* with *HF*,
∴ ∠ *BGC* will coincide with and be equal to ∠ *EHF*.
Again, ∵ ∠ *BAC*=half of ∠ *BGC*, III. 20.
and ∠ *EDF*=half of ∠ *EHF*, III. 20.
∴ ∠ *BAC*= ∠ *EDF*. I. Ax. 7.

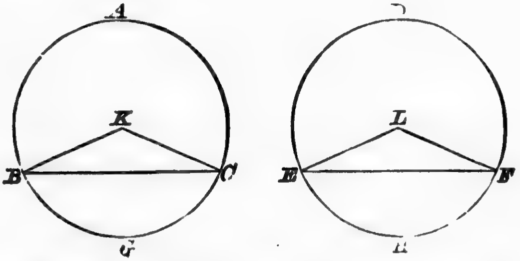
Q. E. D.

Ex. 1. If, in a circle, *AB*, *CD* be two arcs of given magnitude, and *AC*, *BD* be joined to meet in *E*, shew that the angle *AEB* is invariable.

Ex. 2. The straight lines joining the extremities of the chords of two equal arcs of the same circle, towards the same parts, are parallel to each other.

Proposition XXVIII. Theorem.

In equal circles, the arcs, which are subtended by equal chords, must be equal, the greater to the greater, and the less to the less.



Let ABC, DEF be equal circles, and BC, EF equal chords, subtending the major arcs BAC, EDF,

and the minor arcs BGC, EHF.

Then must arc $BAC =$ *arc* EDF, *and arc* $BGC =$ *arc* EHF.

Take the centres K, L, and join KB, KC, LE, LF.

Then $\because KB=LE$, and $KC=LF$, and $BC=EF$,

$\therefore \angle BKC = \angle ELF$. I. c.

Hence, if $\odot$ ABC be applied to $\odot$ DEF,

so that K coincides with L, and KB falls on LE,

then $\because \angle BKC = \angle ELF$, $\therefore KC$ will fall on LF;

and $\because KC = LF$, $\therefore C$ will coincide with F.

Then $\because B$ coincides with E, and C with F,

$\therefore$ arc BAC will coincide with and be equal to arc EDF,

and arc BGC..EHF.

Q. E. D.

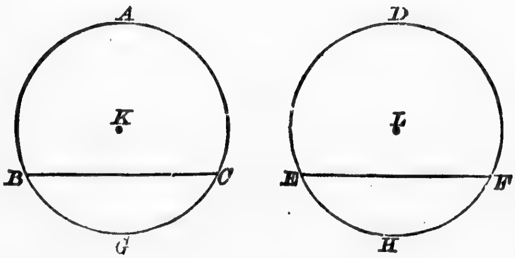
Ex. 1. If, in a circle $ABCD$, the chord AB be equal to the chord DC, AD must be parallel to BC.

Ex. 2. If a straight line, drawn from A the middle point of an arc BC, touch the circle, shew that it is parallel to the chord BC.

Ex. 3. If two equal chords, in a given circle, cut one another, the segments of the one shall be equal to the segments of the other, each to each.

Proposition XXIX. Theorem.

In equal circles, the chords, which subtend equal arcs, must be equal.



Let ABC, DEF be equal circles, and let BC, EF be chords subtending the equal arcs BGC, EHF.

Then must chord BC = chord EF.

Take the centres K, L.

Then, if ⊙ ABC be applied to ⊙ DEF,

so that K coincides with L, and B with E,

and arc BGC falls on arc EHF,

$\because$ arc BGC=arc EHF, $\therefore$ C will coincide with F.

Then $\because$ B coincides with E and C with F,

$\therefore$ chord BC must coincide with and be equal to chord EF.

Q. E. D.

Ex. 1. The two straight lines in a circle, which join the extremities of two parallel chords, are equal to one another.

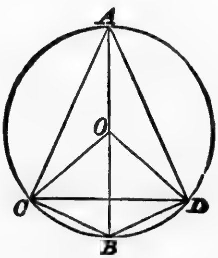
Ex. 2. If three equal chords of a circle, cut one another in the same point, within the circle, that point is the centre.

Note 4. *On the Symmetrical properties of the Circle with regard to its diameter.*

The brief remarks on Symmetry in pp. 107, 108 may now be extended in the following way:

A *figure* is said to be symmetrical with regard to a line, when every perpendicular to the line meets the figure at points which are equidistant from the line.

Hence a Circle is Symmetrical with regard to its Diameter, because the diameter *bisects* every chord, to which it is perpendicular.



Further, suppose AB to be a diameter of the circle $ACBD$, of which O is the centre, and CD to be a chord perpendicular to AB.

Then, if lines be drawn as in the diagram, we know that AB bisects

(1.) The chord CD, III. 1.

(2.) The arcs CAD and CBD, III. 26.

(3.) The angles CAD, COD, CBD, and the reflex angle DOC. I. 4.

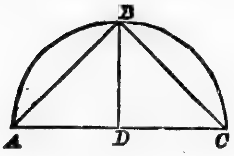
Also, chord CB = chord DB, I. 4.

and chord AC = chord AD. I. 4.

These Symmetrical relations should be carefully observed, because they are often suggestive of methods for the solution of problems.

Proposition XXX. Problem.

To bisect a given arc.



Let ABC be the given arc.

It is required to bisect the arc ABC.

Join AC, and bisect the chord AC in D. I. 10.

From D draw $DB \perp$ to AC. I. 11.

Then will the arc ABC be bisected in B.

Join BA, BC.

Then, in $\triangle$s ADB, CDB,

$\because AD = CD$, and DB is common, and $\angle ADB = \angle CDB$,

$\therefore BA = BC$. I. 4.

But, in the same circle, the arcs, which are subtended by equal chords, are equal, the greater to the greater and the less to the less; III. 28.

and $\because BD$, if produced, is a diameter,

$\therefore$ each of the arcs BA, BC, is less than a semicircle,

and $\therefore$ arc BA = arc BC.

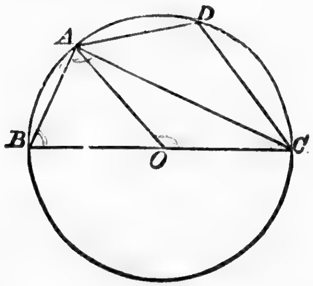
Thus the arc ABC is bisected in B.

Q. E. F.

Ex. If, from any point in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, and the other at right angles to the diameter, the squares on these two lines are together double of the square on the radius.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; and the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.



Let ABC be a $\odot$, O its centre, and BC a diameter.

Draw AC, dividing the $\odot$ into the segments ABC, ADC.

Join BA, AD, DC, AO.

Then must the $\angle$ *in the semicircle* BAC *be a rt.* $\angle$, *and* $\angle$ *in segment* ABC, *greater than a semicircle, less than a rt.* $\angle$, *and* $\angle$ *in segment* ADC, *less than a semicircle, greater than a rt.* $\angle$.

First, $\because BO = AO$, $\therefore \angle BAO = \angle ABO$; I. A.

$\therefore \angle COA =$ twice $\angle BAO$; I. 32.

and $\because CO = AO$, $\therefore \angle CAO = \angle ACO$; I. A.

$\therefore \angle BOA =$ twice $\angle CAO$; I. 32.

$\therefore$ sum of $\angle$s COA, $BOA =$ twice sum of $\angle$s BAO, CAO, that is, two right angles $=$ twice $\angle BAC$.

$\therefore \angle BAC$ is a right angle.

Next, $\because \angle BAC$ is a rt. $\angle$,

$\therefore \angle ABC$ is less than a rt. $\angle$. I. 17.

Lastly, $\because$ sum of $\angle$s ABC, $ADC =$ two rt. $\angle$s, III. 22.

and $\angle ABC$ is less than a rt. $\angle$,

$\therefore \angle ADC$ is greater than a rt. $\angle$. Q. E. D.

NOTE.—For a simpler proof see page 178.

Ex. 1. If a circle be described on the radius of another circle as diameter, any straight line, drawn from the point, where they meet, to the outer circumference, is bisected by the interior one

Ex. 2. If a straight line be drawn to touch a circle, and be parallel to a chord, the point of contact will be the middle point of the arc cut off by the chord.

Ex. 3. If, from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double of the angle contained by the line joining the points of contact, and the diameter drawn through one of them.

Ex. 4. The vertical angle of any oblique-angled triangle inscribed in a circle is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.

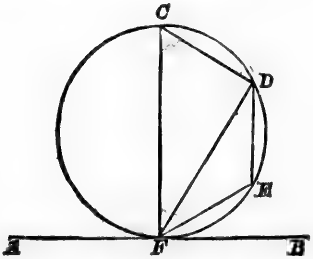
Ex. 5. If, from the extremities of any diameter of a given circle, perpendiculars be drawn to any chord of the circle that is not parallel to the diameter, the less perpendicular shall be equal to that segment of the greater, which is contained between the circumference and the chord.

Ex. 6. If two circles cut one another, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection lie in the same straight line.

Ex. 7. Draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater may be twice the part intercepted by the circumference of the less.

Proposition XXXII. Theorem.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle must be equal to the angles, which are in the alternate segments of the circle.



Let the st. line AB touch the ⊙ $CDEF$ in F.

Draw the chord FD, dividing the ⊙ into segments FCD, FED.

Then must $\angle\, DFB = \angle$ *in segment* FCD,

and $\angle\, DFA = \angle$ *in segment* FED.

From F draw the chord $FC \perp$ to AB.

Then FC is a diameter of the ⊙. III. 19.

Take any pt. E in the arc FED, and join FE, ED, DC.

Then $\because$ FDC is a semicircle, $\therefore \angle\, FDC$ is a rt. $\angle$; III. 31.

$\therefore$ sum of $\angle$ s FCD, CFD = a rt. $\angle$. I. 32.

Also, sum of $\angle$ s DFB, CFD = a rt. $\angle$.

$\therefore$ sum of $\angle$ s DFB, CFD = sum of $\angle$ s FCD, CFD,

and $\therefore \angle\, DFB = \angle\, FCD$,

that is, $\angle\, DFB = \angle$ in segment FCD.

Again, $\because$ $CDEF$ is a quadrilateral fig. inscribed in a ⊙,

$\therefore$ sum of $\angle$ s FED, FCD = two rt. $\angle$ s. III. 22.

Also, sum of $\angle$ s DFA, DFB = two rt. $\angle$ s. I. 13.

$\therefore$ sum of $\angle$ s DFA, DFB = sum of $\angle$ s FED, FCD;

and $\angle\, DFB$ has been proved $= \angle\, FCD$;

$\therefore \angle\, DFA = \angle\, FED$,

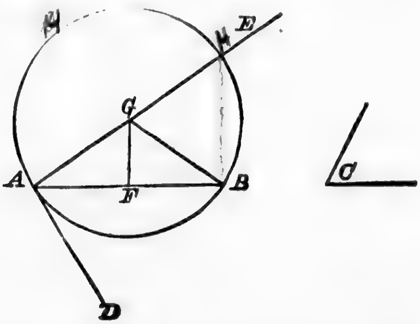
that is, $\angle\, DFA = \angle$ in segment FED.

Q. E. D.

Ex. The chord joining the points of contact of parallel tangents is a diameter.

Proposition XXXIII. Problem.

On a given straight line to describe a segment of a circle capable of containing an angle equal to a given angle.



Let AB be the given st. line, and C the given $\angle$.

It is required to describe on AB a segment of a $\odot$ which shall contain an $\angle = \angle\ C$.

At pt. A in st. line AB make $\angle\ BAD = \angle\ C$. I. 23.

Draw $AE \perp$ to AD, and bisect AB in F.

From F draw $FG \perp$ to AB, meeting AE in G. Join GB.

Then in $\triangle$s AGF, BGF;

$\because AF = BF$, and FG is common, and $\angle\ AFG = \angle\ BFG$;

$\therefore GA = GB$. I. 4.

With G as centre and GA as radius describe a $\odot$ ABH.

Then will AHB be the segment reqd.

For $\because AD$ is $\perp$ to AE, a line passing through the centre,

$\therefore AD$ is a tangent to the $\odot$ ABH. III. 16.

And $\because$ the chord AB is drawn from the pt. of contact A,

$\therefore \angle\ BAD = \angle$ in segment AHB, III. 32.

that is, the segment AHB contains an $\angle = \angle\ C$,

and it is described on AB, as was reqd.

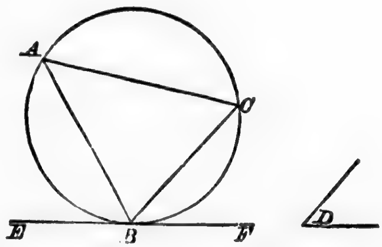
Q. E. F.

Ex. 1. Two circles intersect in A, and through A is drawn a straight line meeting the circles again in P, Q. Prove that the angle between the tangents at P and Q is equal to the angle between the tangents at A.

Ex. 2. From two given points on the same side of a straight line, given in position, draw two straight lines which shall contain a given angle, and be terminated in the given line.

Proposition XXXIV. Problem.

To cut off a segment from a given circle, capable of containing an angle equal to a given angle.



Let ABC be the given ⊙, and D the given ∠.

It is required to cut off from ⊙ ABC a segment capable of containing an $\angle = \angle D$.

Draw the st. line EBF to touch the circle at B.

At B make $\angle FBC = \angle D$.

Then ∵ the chord BC is drawn from the pt. of contact B,

∴ $\angle FBC = \angle$ in segment BAC, III. 32.

that is, the segment BAC contains an $\angle = \angle D$;

and ∴ a segment has been cut off from the ⊙, as was reqd.

Q. E. F.

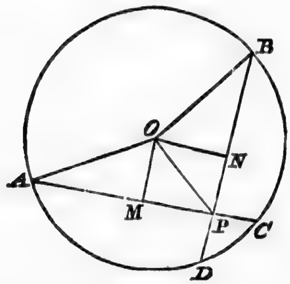
Ex. 1. If two circles touch internally at a point, any straight line passing through the point will divide the circles into segments, capable of containing equal angles.

Ex. 2. Given a side of a triangle, its vertical angle, and the radius of the circumscribing circle: construct the triangle.

Ex. 3. Given the base, vertical angle, and the perpendicular from the extremity of the base on the opposite side: construct the triangle.

PROPOSITION XXXV. THEOREM.

If two chords in a circle cut one another, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.



Let the chords AC, BD in the ⊙ $ABCD$ intersect in the pt. P.

Then must rect. AP, PC=*rect.* BP, PD.

From O, the centre, draw OM, ON ⊥s to AC, BD, and join OA, OB, OP.

Then ∵ AC is divided equally in M and unequally in P,

∴ rect. AP, PC with sq. on MP=sq. on AM. II. 5.

Adding to each the sq. on MO,

rect. AP, PC with sqq. on MP, MO=sqq. on AM, MO;

∴ rect. AP, PC with sq. on OP=sq. on OA. I. 47.

In the same way it may be shewn that

rect. BP, PD with sq. on OP=sq. on OB.

Then ∵ sq. on OA=sq. on OB,

∴ rect. AP, PC with sq. on OP=rect. BP, PD with sq. on OP;

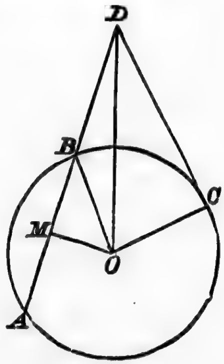
∴ rect. AP. PC=rect. BP, PD. Q. E. D.

Ex. 1. A and B are fixed points, and two circles are described passing through them; PCQ, $P'CQ'$ are chords of these circles intersecting in C, a point in AB; shew that the rectangle CP, CQ is equal to the rectangle CP', CQ'.

Ex. 2. If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

PROPOSITION XXXVI. THEOREM.

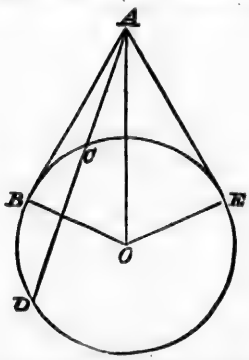
If, from any point without a circle, two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, must be equal to the square on the line which touches it.



Let D be any pt. without the ⊙ ABC,
and let the st. lines DBA, DC be drawn to cut and touch the ⊙.
Then must rect. AD, DB=*sq. on* DC.
From O, the centre, draw OM bisecting AB in M,
and join OB, OC, OD.
Then ∵ AB is bisected in M and produced to D,
∴ rect. AD, DB with sq. on MB=sq. on MD. II. 6.
Adding to each the sq. on MO,
rect. AD, DB with sqq. on MB, MO=sqq. on MD, MO.
Now the angles at M and C are rt. ∠ s; III. 3 and 18.
∴ rect. AD, DB with sq. on OB=sq. on OD;
∴ rect. AD, DB with sq. on OB=sqq. on OC, DC. I. 47.
And sq. on OB=sq. on OC;
∴ rect. AD, DB=sq. on DC. Q. E. D.

Proposition XXXVII. Theorem.

If, from a point without a circle, there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets it, the line which meets must touch the circle.



Let A be a pt. without the ⊙ BCD, of which O is the centre.

From A let two st. lines ACD, AB be drawn, of which ACD cuts the ⊙ and AB meets it.

Then if rect. DA, AC=sq. on AB, AB must touch the ⊙.

Draw AE touching the ⊙ in E, and join OB, OA, OE.

Then ∵ ACD cuts the ⊙, and AE touches it,

∴ rect. DA, AC=sq. on AE. III. 36.

But rect. DA, AC=sq. on AB; Hyp.

∴ sq. on AB=sq. on AE;

∴ $AB=AE$.

Then in the △s OAB, OAE,

∵ $OB=OE$, and OA is common, and $AB=AE$,

∴ ∠ ABO = ∠ AEO. I. c.

But ∠ AEO is a rt. ∠; III. 18.

∴ ∠ ABO is a rt. ∠.

Now BO, if produced, is a diameter of the ⊙;

∴ AB touches the ⊙. III. 16.

Q. E. D.

Miscellaneous Exercises on Book III.

1. The segments, into which a circle is cut by any straight line, contain angles, whose difference is equal to the inclination to each other of the straight lines touching the circle at the extremities of the straight line which divides the circle.

2. If from the point in which a number of circles touch each other, a straight line be drawn cutting all the circles, shew that the lines which join the points of intersection in each circle with its centre will be all parallel.

3. From a point Q in a circle, QN is drawn perpendicular to a chord PP', and QM perpendicular to the tangent at P: shew that the triangles NQP', QPM are equiangular.

4. AB, AC are chords of a circle, and D, E are the middle points of their arcs. If DE be joined, shew that it will cut off equal parts from AB, AC.

5. One angle of a quadrilateral figure inscribed in a circle is a right angle, and from the centre of the circle perpendiculars are drawn to the sides, shew that the sum of their squares is equal to twice the square of the radius.

6. A is the extremity of the diameter of a circle, O any point in the diameter. The chord which is bisected at O subtends a greater or less angle at A than any other chord through O, according as O and A are on the same or opposite sides of the centre.

7. If a straight line in a circle not passing through the centre be bisected by another and this by a third and so on, prove that the points of bisection continually approach the centre of the circle.

8. If a circle be described passing through the opposite angles of a parallelogram, and cutting the four sides, and the points of intersection be joined so as to form a hexagon, the straight lines thus drawn shall be parallel to each other.

9. If two circles touch each other externally and any third circle touch both, prove that the difference of the distances of

the centre of the third circle from the centres of the other two is invariable.

10. Draw two concentric circles, such that those chords of the outer circle, which touch the inner, may equal its diameter.

11. If the sides of a quadrilateral inscribed in a circle be bisected and the middle points of adjacent sides joined, the circles described about the triangles thus formed are all equal and all touch the original circle.

12. Draw a tangent to a circle which shall be parallel to a given finite straight line.

13. Describe a circle, which shall have a given radius, and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.

14. Find a point in the diameter produced of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.

15. Two equal circles intersect in the points A, B, and through B a straight line CBM is drawn cutting them again in C, M. Shew that if with centre C and radius BM a circle be described, it will cut the circle ABC in a point L such that arc AL=arc AB.

Shew also that LB is the tangent at B.

16. AB is any chord and AC a tangent to a circle at A; CDE a line cutting the circle in D and E and parallel to AB. Shew that the triangle ACD is equiangular to the triangle EAB.

17. Two equal circles cut one another in the points A, B; BC is a chord equal to AB; shew that AC is a tangent to the other circle.

18. A, B are two points; with centre B describe a circle, such that its tangent from A shall be equal to a given line.

19. The perpendiculars drawn from the angular points of a triangle to the opposite sides pass through the same point.

20. If perpendiculars be dropped from the angular points of a triangle on the opposite sides, shew that the sum of the squares on the sides of the triangle is equal to twice the sum of the rectangles, contained by the perpendiculars and that part of each intercepted between the angles of the triangles and the point of intersection of the perpendiculars.

21. When two circles intersect, their common chord bisects their common tangent.

22. Two circles intersect in A and B. Two points C and D are taken on one of the circles; CA, CB meet the other circle in E, F, and DA, DB meet it in G, H: shew that FG is parallel to EH.

23. A and B are fixed points, and two circles are described passing through them; CP, CP' are drawn from a point C on AB produced, to touch the circles in P, P'; shew that $CP = CP'$.

24. From each angular point of a triangle a perpendicular is let fall upon the opposite side; prove that the rectangles contained by the segments, into which each perpendicular is divided by the point of intersection of the three, are equal to each other.

25. If from a point without a circle two equal straight lines be drawn to the circumference and produced, shew that they will be at the same distance from the centre.

26. Let O, O' be the centres of two circles which cut each other in A, A'. Let B, B' be two points, taken one on each circumference. Let C, C' be the centres of the circles BAB', $BA'B'$. Then prove that the angle CBC' is the supplement of the angle $OA'O'$.

27. The common chord of two circles is produced to any point P; PA touches one of the circles in A; PBC is any chord of the other: shew that the circle which passes through A, B, C touches the circle to which PA is a tangent.

28. Given the base of a triangle, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base: construct the triangle.

29. If a circle be described about the triangle ABC, and a straight line be drawn bisecting the angle BAC and cutting the circle in D, shew that the angle DCB will be equal to half the angle BAC.

30. If the line AD bisect the angle A in the triangle ABC, and BD be drawn without the triangle making an angle with BC equal to half the angle BAC, shew that a circle may be described about $ABCD$.

31. Two equal circles intersect in A, B: PQT perpendicular to AB meets it in T and the circles in P, Q. AP, BQ meet in R; AQ, BP in S; prove that the angle RTS is bisected by TF.

32. If the angle, contained by any side of a quadrilateral and the adjacent side produced, be equal to the opposite angle of the quadrilateral, prove that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

33. If DE be drawn parallel to the base BC of a triangle ABC, prove that the circles described about the triangles ABC and ADE have a common tangent at A.

34. Describe a square equal to the difference of two given squares.

35. If tangents be drawn to a circle from any point without it, and a third line be drawn between the point and the centre of the circle, touching the circle, the perimeter of the triangle formed by the three tangents will be the same for all positions of the third point of contact.

36. If on the sides of any triangle as chords, circles be described, of which the segments external to the triangle contain angles respectively equal to the angles of a given triangle, those circles will intersect in a point.

37. Prove that if ABC be a triangle inscribed in a circle, such that $BA=BC$, and AA' be drawn parallel to BC, meeting the circle again in A', and $A'B$ be joined cutting AC in E, BA touches the circle described about the triangle AEA'.

38. Describe a circle, cutting the sides of a given square, so that its circumference may be divided at the points of intersection into eight equal arcs.

39. AB is the diameter of a semicircle, D and E any two points on its circumference. Shew that if the chords joining A and B with D and E, either way, intersect in F and G, the tangents at D and E meet in the middle point of the line FG, and that FG produced is at right angles to AB.

40. Shew that the square on the tangent drawn from any point in the outer of two concentric circles to the inner equals the difference of the squares on the tangents, drawn from any point, without both circles, to the circles.

41. If from a point without a circle, two tangents PT, PT', at right angles to one another, be drawn to touch the circle, and if from T any chord TQ be drawn, and from T' a perpendicular $T'M$ be dropped on TQ, then $T'M = QM$.

42. Find the loci:

(1.) Of the centres of circles passing through two given points.

(2.) Of the middle points of a system of parallel chords in a circle.

(3.) Of points such that the difference of the distances of each from two given straight lines is equal to a given straight line.

(4.) Of the centres of circles touching a given line in a given point.

(5.) Of the middle points of chords in a circle that pass through a given point.

(6.) Of the centres of circles of given radius which touch a given circle.

(7.) Of the middle points of chords of equal length in a circle.

(8.) Of the middle points of the straight lines drawn from a given point to meet the circumference of a given circle.

43. If the base and vertical angle of a triangle be given, find the locus of the vertex.

44. A straight line remains parallel to itself while one of its extremities describes a circle. What is the locus of the other extremity?

45. A ladder slips down between a vertical wall and a horizontal plane: what is the locus of its middle point?

46. ABC is a line drawn from a point A, without a circle, to meet the circumference in B and C. Tangents are drawn to the circle at B and C which meet in D. What is the locus of D?

47. The angular points A, C of a parallelogram $ABCD$ move on two fixed straight lines OA, OC, whose inclination is equal to the angle BCD; shew that one of the points B, D, which is the more remote from O, will move on a fixed straight line passing through O.

48. On the line AB is described the segment of a circle in the circumference of which any point C is taken. If AC, BC be joined, and a point P taken in AC so that CP is equal to CB, find the locus of P.

49. The centre of the circle $CBED$ is on the circumference of ABD. If from any point A the lines ABC and AED be drawn to cut the circles, the chord BE is parallel to CD.

50. If a parallelogram be described having the diameter of a given circle for one of its sides, and the intersection of its diagonals on the circumference, shew that the extremity of each of the diagonals moves on the circumference of another circle of double the diameter of the first.

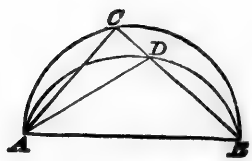
51. One diagonal of a quadrilateral inscribed in a circle is fixed, and the other of constant length. Shew that the sides will meet, if produced, on the circumferences of two fixed circles.

We here insert Euclid's proofs of Props. 23, 24 of Book III. first observing that he gives the following definition of similar segments :—

DEF. *Similar segments of circles are those in which the angles are equal, or which contain equal angles.*

PROPOSITION XXIII. THEOREM.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with each other.



If it be possible, on the same base AB, and on the same side of it, let there be two similar segments of ⊙s, ABC, ABD, which do not coincide.

Because ⊙ ADB cuts ⊙ ACB in pts. A and B, they cannot cut one another in any other pt., and ∴ one of the segments must fall within the other. III (10)

Let ADB fall within ACB.

Draw the st. line BDC and join CA, DA.

Then ∵ segment ADB is similar to segment ACB,

∴ ∠ ADB = ∠ ACB.

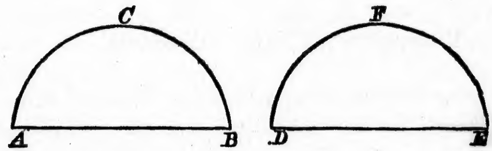
Or the extr. ∠ of a △ = the intr. and opposite ∠, which is impossible ; I (16)

∴ the segments cannot but coincide.

Q. E. D.

Proposition XXIV. Theorem.

Similar segments of circles, upon equal straight lines, are equal to one another.



Let ABC, DEF be similar segments of ⊙s on equal st. lines AB, DE.

Then must segment ABC=*segment* DEF.

For if segment ABC be applied to segment DEF, so that A may be on D and AB on DE, then B will coincide with E, and AB with DE;

∴ segment ABC must also coincide with segment DEF; III. 23.

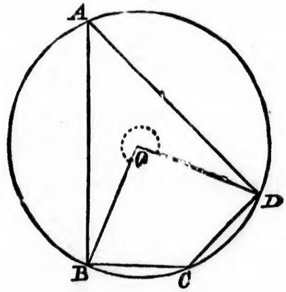
∴ segment ABC=segment DEF. Ax. 8.

Q. E. D.

We gave one Proposition, C, page 150, as an example of the way in which the conceptions of Flat and Reflex Angles may be employed to extend and simplify Euclid's proofs. We here give the proofs, based on the same conceptions, of the important propositions XXII. and XXIII.

Proposition XXII. Theorem.

The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.



Let $ABCD$ be a quadrilateral fig. inscribed in a $\odot$.

Then must each pair of its opposite $\angle$ *s be together equal to two rt.* $\angle$ *s.*

From O, the centre, draw OB, OD.

Then $\because \angle BOD$ = twice $\angle BAD$, III. 20.

and the reflex $\angle DOB$ = twice $\angle BCD$, III. C. p. 150.

$\therefore$ sum of $\angle$ s at O = twice sum of $\angle$ s BAD, BCD.

But sum of $\angle$ s at O = 4 right $\angle$ s; I. 15, Cor. 2.

$\therefore$ twice sum of $\angle$ s BAD, BCD = 4 right $\angle$ s;

$\therefore$ sum of $\angle$ s BAD, BCD = two right $\angle$ s.

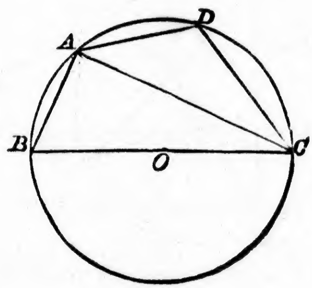
Similarly, it may be shewn that

sum of $\angle$ s ABC, ADC = two right $\angle$ s.

Q. E. D.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; and the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.



Let ABC be a ⊙, of which O is the centre and BC a diameter.

Draw AC, dividing the ⊙ into the segments ABC, ADC.

Join BA, AD, DC.

Then must the ∠ in the semicircle BAC be a rt. ∠, and ∠ in segment ABC, greater than a semicircle, less than a rt. ∠, and ∠ in segment ADC, less than a semicircle, greater than a rt. ∠.

First, ∵ the flat angle BOC=twice ∠ BAC, III. C p. 150.

∴ ∠ BAC is a rt. ∠.

Next, ∵ ∠ BAC is a rt. ∠,

∴ ∠ ABC is less than a rt. ∠. I. 17.

Lastly, ∵ sum of ∠ s ABC, ADC=two rt. ∠ s, III. 22.

and ∠ ABC is less than a rt. ∠,

∴ ∠ ADC is greater than a rt. ∠.

Q. E. D.

Book I page 5-6 (3.) Let aBC be a $\triangle$ aF the bisector of angle at a + aD the perpen from a on BC then $\angle FaD$ = half diff. $\angle$s B + C. At point a in st line Fa make $\angle GaG = \angle FaD$ I (23)

$\angle FaB = \angle Fac$ Hyp.

$\angle FaG = \angle FaD$ Constr

$\therefore$ rem angle GaB = rem $\angle$ Dac

$\angle B + \angle BaD$ = rt $\angle$ I 32

$\angle C + \angle CaD$ = " " I 32

$\therefore \angle B + \angle BaD = \angle C + \angle CaD$ Ax I

$\therefore$ Diff between $\angle B$ + $\angle C$ = diff $\angle DaD$ + $\angle CaD$

= " $\angle BaD$ + $\angle GaB$ (since $\angle GaB = \angle CaD$)

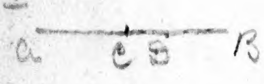
= $\angle GaD = 2 FaD$ (since $\angle FaG = \angle FaD$)

$\therefore \angle FaD = \frac{1}{2}$ diff between $\angle B$ + $\angle C$

Prop's 5 + 6 in 2nd Book may be included in one enunciation:
The rectangle under the sum + diff of 2 st lines is equal to the diff of the sqq described on those st lines
or thus
The rectangle contained by 2 st lines together with the sq described on half of their diff is equal to the sq described on $\frac{1}{2}$ their sum

---

V



$aD.DC + \overline{CD}^2 = \overline{CB}^2$

$aD.DB = \overline{CB}^2 - \overline{CD}^2$

$(aC + CD)(aC - CD) = \overline{aC}^2 - \overline{CD}^2$

VI

$aC.CB + \overline{DB}^2 = \overline{DC}^2$

$aC.CD = \overline{DC}^2 - \overline{DB}^2$

$(DC + aD)(DC - aD) = \overline{DC}^2 - \overline{aD}^2$